

Complex Analysis

Chapter XI. Entire Functions

XI.2. The Genus and Order of an Entire Function—Proofs of Theorems



Theorem XI.2.6

Theorem XI.2.6. Let f be an entire function of genus μ . For each positive number α there is a number r_0 such that for $|z| > r_0$ we have $|f(z)| < \exp(\alpha|z|^{\mu+1})$.

Proof. Since f is an entire function of genus μ then

$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{\mu}(z/a_n)$ where g is a polynomial of degree at most μ . Notice that if $|z| < 1/2$ then

$$\begin{aligned} \log |E_{\mu}(z)| &= \log |(1-z) \exp(z + z^2/2 + \cdots + z^{\mu}/\mu)| \\ &= \operatorname{Re}(\log(1-z) \exp(z + z^2/2 + \cdots + z^{\mu}/\mu)) \text{ since} \\ &\quad \operatorname{Re}(\log z) = \operatorname{Re}(\log(|z|e^{i \arg(z)})) \\ &= \operatorname{Re}(\log |z| + i \arg(z)) = \log |z| \\ &= \operatorname{Re}(\log(1-z) + z + z^2/2 + \cdots + z^{\mu}/\mu) \\ &= \operatorname{Re}\left(-\frac{1}{\mu+1}z^{\mu+1} - \frac{1}{\mu+2}z^{\mu+1} - \cdots\right) \end{aligned}$$

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Proof (continued).

$$\begin{aligned} \text{since } \log(1-z) &= \sum_{n=1}^{\infty} x^n \text{ for } |z| < 1 \\ &\leq |z|^{\mu+1} \left(\frac{1}{\mu+1} + \frac{|z|}{\mu+2} + \frac{|z|^2}{\mu+3} + \cdots \right) \\ &\leq |z|^{\mu+1} \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \right) \text{ since } |z| < 1/2 \text{ and } \mu \geq 0 \\ &= |z|^{\mu+1} \frac{1}{1-1/2} = 2|z|^{\mu+1}. \end{aligned} \quad (2.7)$$

Also

$$\begin{aligned} |E_{\mu}(z)| &= \left| (1-z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^{\mu}}{\mu}\right) \right| \\ &\leq (1+|z|) \exp\left(|z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^{\mu}}{\mu}\right) \end{aligned}$$

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Theorem XI.2.6 (continued 2)

Proof (continued). ... so that

$$\log |E_{\mu}(z)| \leq \log(1+|z|) + |z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^{\mu}}{\mu}.$$

Hence,

$$\lim_{z \rightarrow \infty} \frac{\log |E_{\mu}(z)|}{|z|^{\mu+1}} \leq \lim_{z \rightarrow \infty} \frac{\log(1+|z|) + |z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^{\mu}}{\mu}}{|z|^{\mu+1}} = 0.$$

So for any $A > 0$, there is $R > 0$ such that for $|z| > R$ we have $\frac{\log |E_{\mu}(z)|}{|z|^{\mu+1}} \leq A$ or

$$\log |E_{\mu}(z)| \leq A|z|^{\mu+1}. \quad (2.8)$$

But on the compact set $\{z \mid 1/2 \leq |z| \leq R\}$ the function

$|z|^{\mu+1} \log |E_{\mu}(z)|$ is continuous except at $z = +1$ where it tends to $-\infty$. So $|z|^{\mu+1} \log |E_{\mu}(z)|$ is bounded above on $\{z \mid 1/2 \leq |z| \leq R\}$, say by $B > 0$.

Proof (continued). Then

$$\log |E_\mu(z)| \leq B|z|^{\mu+1} \text{ for } 1/2 \leq |z| \leq R. \quad (2.9)$$

So we have a bound on $\log |E_\mu(z)|$ on these sets by (2.7), (2.8), and (2.9) and can conclude

$$\log |E_\mu(z)| \leq M|z|^{\mu+1} \text{ for all } z \in \mathbb{C} \quad (2.10)$$

where $M = \max\{2, A, B\}$.

Since f has finite rank then $\sum_{n=1}^{\infty} |a_n|^{-(\mu+1)} < \infty$ so for a given $\alpha > 0$, there is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |a_n|^{-(\mu+1)} < \frac{\alpha}{4M}$$

(since the tail of a convergent series can be made arbitrarily small). Then from (2.10)

$$\sum_{n=N+1}^{\infty} \log |E_\mu(a/a_n)| \leq M \sum_{n=N+1}^{\infty} \left| \frac{z}{a_n} \right|^{\mu+1} \leq \frac{\alpha}{4} |z|^{\mu+1} \text{ for all } z \in \mathbb{C}. \quad (2.11)$$

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Theorem XI.2.6

Proof (continued). Since g is a polynomial of degree at most μ (since f is of genus μ) then $\lim_{z \rightarrow \infty} \frac{|g(z)|}{|z|^{\mu+1}} = 0$ and since $\lim_{z \rightarrow \infty} \frac{\log |z|}{|z|^{\mu+1}} = 0$ then $\lim_{z \rightarrow \infty} \frac{m \log |z| + |g(z)|}{|z|^{\mu+1}} = 0$. So there is $r_3 > 0$ such that $\frac{m \log |z| + |g(z)|}{|z|^{\mu+1}} < \frac{\alpha}{2}$ for $|z| > r_3$. We then have

$$\begin{aligned} \log |f(z)| &= \log \left| z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n) \right| \\ &= m \log |z| + |g(z)| + \sum_{n=1}^{\infty} \log |E_\mu(z/a_n)| \\ &< \frac{\alpha}{2} |z|^{\mu+1} + \frac{\alpha}{2} |z|^{\mu+1} \text{ for } |z| > r_0 = \max\{r_2, r_3\} \text{ by (2.12)} \\ &= \alpha |z|^{\mu+1}. \end{aligned}$$

So $e^{\log |f(z)|} < e^{\alpha |z|^{\mu+1}}$ or $|f(z)| < e^{\alpha |z|^{\mu+1}}$ for $|z| > r_0$, as claimed. \square

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Proof (continued). Since parameter A in (2.8) can be chosen as small as desired (though this effects the value of R), choose $r_1 > 0$ such that

$$\log |E_\mu(z)| \leq \frac{\alpha}{4N} |z|^{\mu+1} \text{ for } |z| > r_1$$

(that is, replace α with α/N in (2.11)). With $r_2 = \max\{|a_1|/r_1, |a_2|/r_1, \dots, |a_N|/r_1\}$ we have

$$\sum_{n=1}^{\infty} \log |E_\mu(z/a_n)| \leq \frac{\alpha}{4} |z|^{\mu+1} \text{ for } |z| > r_2$$

(replacing x in the equation above, where we had $|z| > r_1$, with z/z_n here where we have $|z/a_n| > r_1$ or $|z| > |z_n|/r_1 \geq r_2$). Combining this with (2.11) gives

$$\begin{aligned} \log P(z) &= \log \left(\prod_{n=1}^{\infty} E_\mu \left(\frac{z}{z_n} \right) \right) = \sum_{n=1}^{\infty} \log \left(E_\mu \left(\frac{z}{a_n} \right) \right) \\ &\leq \frac{\alpha}{4} |z|^{\mu+1} + \frac{\alpha}{4} |z|^{\mu+1} = \frac{\alpha}{2} |z|^{\mu+1} \text{ for } |z| > r_2. \end{aligned} \quad (2.12)$$

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Theorem XI.2.4

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Proposition XI.2.14. Let f be an entire function of finite order λ . If $\varepsilon > 0$ then $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$ for all z with $|z|$ sufficiently large, and z can be found with $|z|$ as large as desired such that $|f(z)| \geq \exp(|z|^{\lambda+\varepsilon})$.

Proof. Since e^{x^λ} is an increasing function, then for any nonzero $z \in \mathbb{C}$, if $a < b$ then $\exp(|z|^a) < \exp(|z|^b)$. For $\varepsilon > 0$, $\lambda + \varepsilon > \lambda$ so, since $\lambda = \inf\{a \mid |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\}$, we have for $|z|$ sufficiently large that $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$. Since $\lambda - \varepsilon < \lambda$ then $\lambda - \varepsilon \notin \{a \mid |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\}$. So for all $r > 0$ there is $z_{r,\varepsilon} \in \mathbb{C}$ with $|z_{r,\varepsilon}| > r$ and $|f(z_{r,\varepsilon})| \geq \exp(|z_{r,\varepsilon}|^{\lambda-\varepsilon})$, as claimed. \square

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