Complex Analysis

Chapter XI. Entire Functions

XI.2. The Genus and Order of an Entire Function—Proofs of Theorems

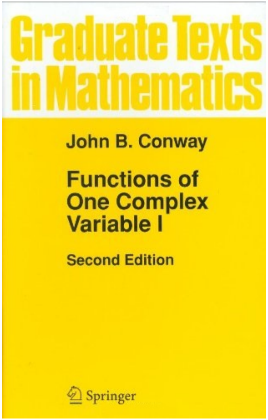

Table of contents

Theorem XI.2.6

Theorem XI.2.6. Let f be an entire function of genus μ. For each positive number α there is a number r_0 such that for $|z| > r_0$ we have $|f(z)| < \exp(\alpha|z|^{\mu+1})$.

Proof. Since f is an entire function of genus μ then $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n)$ where g is a polynomial of degree at most μ.

Theorem XI.2.6

Theorem XI.2.6. Let f be an entire function of genus μ. For each positive number α there is a number r_0 such that for $|z| > r_0$ we have $|f(z)| < \exp(\alpha|z|^{\mu+1})$.

Proof. Since f is an entire function of genus μ then $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n)$ where g is a polynomial of degree at most μ. Notice that if $|z| < 1/2$ then

$$\begin{aligned}
\log|E_\mu(z)| &= \log|(1-z)\exp(z+z^2/2+\cdots+z^\mu/\mu)| \\
&= \text{Re}(\log(1-z)\exp(z+z^2/2+\cdots+z^\mu/\mu)) \text{ since} \\
&\quad \text{Re}(\log z) = \text{Re}(\log(|z|e^{i\arg(z)})) \\
&\quad = \text{Re}(\log|z| + i\arg(z)) = \log|z| \\
&= \text{Re}(\log(1-z) + z + z^2/2 + \cdots z^\mu/\mu) \\
&= \text{Re}\left(-\frac{1}{\mu+1}z^{\mu+1} - \frac{1}{\mu+2}z^{\mu+1} - \cdots\right)
\end{aligned}$$

Theorem XI.2.6

Theorem XI.2.6. Let f be an entire function of genus μ. For each positive number α there is a number r_0 such that for $|z| > r_0$ we have $|f(z)| < \exp(\alpha|z|^{\mu+1})$.

Proof. Since f is an entire function of genus μ then $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n)$ where g is a polynomial of degree at most μ. Notice that if $|z| < 1/2$ then

$$\begin{aligned}
\log|E_\mu(z)| &= \log|(1-z)\exp(z + z^2/2 + \cdots + z^\mu/\mu)| \\
&= \text{Re}(\log(1-z)\exp(z + z^2/2 + \cdots + z^\mu/\mu)) \text{ since} \\
&\quad \text{Re}(\log z) = \text{Re}(\log(|z|e^{i\arg(z)})) \\
&\quad = \text{Re}(\log|z| + i\arg(z)) = \log|z| \\
&= \text{Re}(\log(1-z) + z + z^2/2 + \cdots z^\mu/\mu) \\
&= \text{Re}\left(-\frac{1}{\mu+1}z^{\mu+1} - \frac{1}{\mu+2}z^{\mu+1} - \cdots\right)
\end{aligned}$$

Proof (continued).

$$\text{since } \log(1-z) = \sum_{n=1}^{\infty} x^n \text{ for } |z| < 1$$

$$\begin{aligned}
&\leq |z|^{\mu+1}\left(\frac{1}{\mu+1} + \frac{|z|}{\mu+2} + \frac{|z|^2}{\mu+3} + \cdots\right) \\
&\leq |z|^{\mu+1}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right) \text{ since } |z| < 1/2 \text{ and } \mu \geq 0 \\
&= |z|^{\mu+1}\frac{1}{1-1/2} = 2|z|^{\mu+1}. \qquad (2.7)
\end{aligned}$$

Also

$$|E_\mu(z)| = \left|(1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^\mu}{\mu}\right)\right|$$

$$\leq (1+|z|)\exp\left(|z| + \frac{|z|^2}{2} + \cdots \frac{|z|^\mu}{\mu}\right)$$

Proof (continued).

$$\text{since } \log(1-z) = \sum_{n=1}^{\infty} x^n \text{ for } |z| < 1$$

$$\begin{aligned}
&\leq |z|^{\mu+1}\left(\frac{1}{\mu+1} + \frac{|z|}{\mu+2} + \frac{|z|^2}{\mu+3} + \cdots\right) \\
&\leq |z|^{\mu+1}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right) \text{ since } |z| < 1/2 \text{ and } \mu \geq 0 \\
&= |z|^{\mu+1}\frac{1}{1-1/2} = 2|z|^{\mu+1}. \qquad (2.7)
\end{aligned}$$

Also

$$|E_\mu(z)| = \left|(1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^\mu}{\mu}\right)\right|$$

$$\leq (1+|z|)\exp\left(|z| + \frac{|z|^2}{2} + \cdots \frac{|z|^\mu}{\mu}\right)$$

Theorem XI.2.6 (continued 2)

Proof (continued). ... so that

$$\log|E_\mu(z)| \leq \log(1+|z|) + |z| + \frac{|z|^2}{z} + \cdots + \frac{|z|^\mu}{\mu}.$$

Hence,

$$\lim_{z\to\infty} \frac{\log|E_\mu(z)|}{|z|^{\mu+1}} \leq \lim_{z\to\infty} \frac{\log(1+|z|) + |z| + |z|^2/2 + \cdots + |z|^\mu/\mu}{|z|^{\mu+1}} = 0.$$

So for any $A > 0$, there is $R > 0$ such that for $|z| > R$ we have $\frac{\log|E_\mu(z)|}{|z|^{\mu+1}} \leq A$ or

$$\log|E_\mu(z)| \leq A|z|^{\mu+1}. \qquad (2.8)$$

Theorem XI.2.6 (continued 2)

Proof (continued). ...so that

$$\log|E_\mu(z)| \leq \log(1+|z|) + |z| + \frac{|z|^2}{z} + \cdots + \frac{|z|^\mu}{\mu}.$$

Hence,

$$\lim_{z\to\infty} \frac{\log|E_\mu(z)|}{|z|^{\mu+1}} \leq \lim_{z\to\infty} \frac{\log(1+|z|) + |z| + |z|^2/2 + \cdots + |z|^\mu/\mu}{|z|^{\mu+1}} = 0.$$

So for any $A > 0$, there is $R > 0$ such that for $|z| > R$ we have $\frac{\log|E_\mu(z)|}{|z|^{\mu+1}} \leq A$ or

$$\log|E_\mu(z)| \leq A|z|^{\mu+1}. \qquad (2.8)$$

But on the compact set $\{z \mid 1/2 \leq |z| \leq R\}$ the function $|z|^{\mu+1}\log|E_\mu(z)|$ is continuous except at $z = +1$ where it tens to $-\infty$. So $|z|^{\mu+1}\log|E_\mu(z)|$ is bounded above on $\{z \mid 1/2 \leq |z| \leq R\}$, say by $B > 0$.

Theorem XI.2.6 (continued 2)

Proof (continued). . . . so that

$$\log|E_\mu(z)| \leq \log(1+|z|) + |z| + \frac{|z|^2}{z} + \cdots + \frac{|z|^\mu}{\mu}.$$

Hence,

$$\lim_{z\to\infty} \frac{\log|E_\mu(z)|}{|z|^{\mu+1}} \leq \lim_{z\to\infty} \frac{\log(1+|z|)+|z|+|z|^2/2+\cdots+|z|^\mu/\mu}{|z|^{\mu+1}} = 0.$$

So for any $A > 0$, there is $R > 0$ such that for $|z| > R$ we have $\frac{\log|E_\mu(z)|}{|z|^{\mu+1}} \leq A$ or

$$\log|E_\mu(z)| \leq A|z|^{\mu+1}. \qquad (2.8)$$

But on the compact set $\{z \mid 1/2 \leq |z| \leq R\}$ the function $|z|^{\mu+1}\log|E_\mu(z)|$ is continuous except at $z = +1$ where it tens to $-\infty$. So $|z|^{\mu+1}\log|E_\mu(z)|$ is bounded above on $\{z \mid 1/2 \leq |z| \leq R\}$, say by $B > 0$.

Proof (continued). Then

$$\log|E_\mu(z)| \le B|z|^{\mu+1} \text{ for } 1/2 \le |z| \le R. \qquad (2.9)$$

So we have a bound on $\log|E_\mu(z)|$ on these sets by (2.7), (2.8), and (2.9) and can conclude

$$\log|E_\mu(z)| \le M|z|^{\mu+1} \text{ for all } z \in \mathbb{C} \qquad (2.10)$$

where $M = \max\{2, A, B\}$.

Since f has finite rank then $\sum_{n=1}^{\infty} |a_n|^{-(\mu+1)} < \infty$ so for a given $\alpha > 0$, there is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |a_n|^{-(\mu+1)} < \frac{\alpha}{4M}$$

(since the tail of a convergent series can be made arbitrarily small).

Proof (continued). Then

$$\log|E_\mu(z)| \le B|z|^{\mu+1} \text{ for } 1/2 \le |z| \le R. \qquad (2.9)$$

So we have a bound on $\log|E_\mu(z)|$ on these sets by (2.7), (2.8), and (2.9) and can conclude

$$\log|E_\mu(z)| \le M|z|^{\mu+1} \text{ for all } z \in \mathbb{C} \qquad (2.10)$$

where $M = \max\{2, A, B\}$.

Since f has finite rank then $\sum_{n=1}^\infty |a_n|^{-(\mu+1)} < \infty$ so for a given $\alpha > 0$, there is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^\infty |a_n|^{-(\mu+1)} < \frac{\alpha}{4M}$$

(since the tail of a convergent series can be made arbitrarily small). Then from (2.10)

$$\sum_{n=N+1}^\infty \log|E_\mu(a/a_n)| \le M \sum_{n=N+1}^\infty \left|\frac{z}{a_n}\right|^{\mu+1} \le \frac{\alpha}{4}|z|^{\mu+1} \text{ for all } z \in \mathbb{C}. \qquad (2.11)$$

Proof (continued). Then

$$\log|E_\mu(z)| \le B|z|^{\mu+1} \text{ for } 1/2 \le |z| \le R. \qquad (2.9)$$

So we have a bound on $\log|E_\mu(z)|$ on these sets by (2.7), (2.8), and (2.9) and can conclude

$$\log|E_\mu(z)| \le M|z|^{\mu+1} \text{ for all } z \in \mathbb{C} \qquad (2.10)$$

where $M = \max\{2, A, B\}$.

Since f has finite rank then $\sum_{n=1}^\infty |a_n|^{-(\mu+1)} < \infty$ so for a given $\alpha > 0$, there is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^\infty |a_n|^{-(\mu+1)} < \frac{\alpha}{4M}$$

(since the tail of a convergent series can be made arbitrarily small). Then from (2.10)

$$\sum_{n=N+1}^\infty \log|E_\mu(a/a_n)| \le M \sum_{n=N+1}^\infty \left|\frac{z}{a_n}\right|^{\mu+1} \le \frac{\alpha}{4}|z|^{\mu+1} \text{ for all } z \in \mathbb{C}. \qquad (2.11)$$

Proof (continued). Since parameter A in (2.8) can be chosen as small as desired (though this effects the value of R), choose $r_1 > 0$ such that

$$\log|E_\mu(z)| \leq \frac{\alpha}{4N}|z|^{\mu+1} \text{ for } |z| > r_1$$

(that is, replace α with α/N in (2.11)). With $r_2 = \max\{|a_1|r_n, |a_2|r_1, \ldots, |a_N|r_1\}$ we have

$$\sum_{n=1}^{\infty} \log|E_\mu(z/a_n)| \leq \frac{\alpha}{4}|z|^{\mu+1} \text{ for } |z| > r_2$$

(replacing x in the equation above, where we had $|z| > r_1$, with z/z_n here where we have $|z/a_n| > r_1$ or $|z| > |z_n|r_1 \geq r_2$). Combining this with (2.11) gives

$$\begin{aligned} \log P(z) &= \log\left(\prod_{n=1}^{\infty} E_\mu\left(\frac{z}{z_n}\right)\right) = \sum_{n=1}^{\infty} \log\left(E_\mu\left(\frac{z}{a_n}\right)\right) \\ &\leq \frac{\alpha}{4}|z|^{\mu+1} + \frac{\alpha}{4}|z|^{\mu+1} = \frac{\alpha}{2}|z|^{\mu+1} \text{ for } |z| > r_2. \qquad (2.12) \end{aligned}$$

Proof (continued). Since parameter A in (2.8) can be chosen as small as desired (though this effects the value of R), choose $r_1 > 0$ such that

$$\log|E_\mu(z)| \leq \frac{\alpha}{4N}|z|^{\mu+1} \text{ for } |z| > r_1$$

(that is, replace α with α/N in (2.11)). With $r_2 = \max\{|a_1|r_n, |a_2|r_1, \ldots, |a_N|r_1\}$ we have

$$\sum_{n=1}^{\infty} \log|E_\mu(z/a_n)| \leq \frac{\alpha}{4}|z|^{\mu+1} \text{ for } |z| > r_2$$

(replacing x in the equation above, where we had $|z| > r_1$, with z/z_n here where we have $|z/a_n| > r_1$ or $|z| > |z_n|r_1 \geq r_2$). Combining this with (2.11) gives

$$\begin{aligned} \log P(z) &= \log\left(\prod_{n=1}^{\infty} E_\mu\left(\frac{z}{z_n}\right)\right) = \sum_{n=1}^{\infty} \log\left(E_\mu\left(\frac{z}{a_n}\right)\right) \\ &\leq \frac{\alpha}{4}|z|^{\mu+1} + \frac{\alpha}{4}|z|^{\mu+1} = \frac{\alpha}{2}|z|^{\mu+1} \text{ for } |z| > r_2. \qquad (2.12) \end{aligned}$$

Proof (continued). Since g is a polynomial of degree at most μ (since f is of genus μ) then $\lim_{z\to\infty} \frac{|g(z)|}{|z|^{\mu+1}} = 0$ and since $\lim_{z\to\infty} \frac{\log|z|}{|z|^{\mu+1}} = 0$ then $\lim_{z\to\infty} \frac{m\log|z|+|g(z)|}{|z|^{\mu+1}} = 0$. So there is $r_3 > 0$ such that $\frac{m\log|z|+|g(z)|}{|z|^{\mu+1}} < \frac{\alpha}{2}$ for $|z| > r_3$. We then have

$$\begin{aligned}
\log|f(z)| &= \log\left|z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n)\right| \\
&= m\log|z| + |g(z)| + \sum_{n=1}^{\infty} \log|E_\mu(z/a_n)| \\
&< \frac{\alpha}{2}|z|^{\mu+1} + \frac{\alpha}{2}|z|^{\mu+1} \text{ for } |z| > r_0 = \max\{r_2, r_3\} \text{ by (2.12)} \\
&= \alpha|z|^{\mu+1}.
\end{aligned}$$

So $e^{\log|f(z)|} < e^{\alpha|z|^{\mu+1}}$ or $|f(z)| < e^{\alpha|z|^{\mu+1}}$ for $|z| > r_0$, as claimed. □

Proof (continued). Since g is a polynomial of degree at most μ (since f is of genus μ) then $\lim_{z\to\infty} \frac{|g(z)|}{|z|^{\mu+1}} = 0$ and since $\lim_{z\to\infty} \frac{\log|z|}{|z|^{\mu+1}} = 0$ then $\lim_{z\to\infty} \frac{m\log|z|+|g(z)|}{|z|^{\mu+1}} = 0$. So there is $r_3 > 0$ such that $\frac{m\log|z|+|g(z)|}{|z|^{\mu+1}} < \frac{\alpha}{2}$ for $|z| > r_3$. We then have

$$\begin{aligned}
\log|f(z)| &= \log\left|z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu(z/a_n)\right| \\
&= m\log|z| + |g(z)| + \sum_{n=1}^{\infty} \log|E_\mu(z/a_n)| \\
&< \frac{\alpha}{2}|z|^{\mu+1} + \frac{\alpha}{2}|z|^{\mu+1} \text{ for } |z| > r_0 = \max\{r_2, r_3\} \text{ by (2.12)} \\
&= \alpha|z|^{\mu+1}.
\end{aligned}$$

So $e^{\log|f(z)|} < e^{\alpha|z|^{\mu+1}}$ or $|f(z)| < e^{\alpha|z|^{\mu+1}}$ for $|z| > r_0$, as claimed. □

Theorem XI.2.4

Proposition XI.2.14. Let f be an entire function of finite order λ. If $\varepsilon > 0$ then $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$ for all z with $|z|$ sufficiently large, and z can be found with $|z|$ as large as desired such that $|f(z)| \geq \exp(|z|^{\lambda+\varepsilon})$.

Proof. Since e^x is an increasing function, then for any nonzero $z \in \mathbb{C}$, if $a < b$ then $\exp(|z|^a) < \exp(|z|^b)$. For $\varepsilon > 0$, $\lambda + \varepsilon > \lambda$ so, since $\lambda = \inf\{a \mid |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\}$, we have for $|z|$ sufficiently large that $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$.

Theorem XI.2.4

Proposition XI.2.14. Let f be an entire function of finite order λ. If $\varepsilon > 0$ then $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$ for all z with $|z|$ sufficiently large, and z can be found with $|z|$ as large as desired such that $|f(z)| \geq \exp(|z|^{\lambda+\varepsilon})$.

Proof. Since e^x is an increasing function, then for any nonzero $z \in \mathbb{C}$, if $a < b$ then $\exp(|z|^a) < \exp(|z|^b)$. For $\varepsilon > 0$, $\lambda + \varepsilon > \lambda$ so, since $\lambda = \inf\{a \mid |f(z)| < \exp(|z|^a)$ for $|z|$ sufficiently large$\}$, we have for $|z|$ sufficiently large that $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$. Since $\lambda - \varepsilon < \lambda$ then $\lambda - \varepsilon \not\in \{a \mid |f(z)| < \exp(|z|^a)$ for $|z|$ sufficiently large$\}$. So for all $r > 0$ there is $z_{r,\varepsilon} \in \mathbb{C}$ with $|z_{r,\varepsilon}| > r$ and $|f(z_{r,,\varepsilon})| \geq \exp(|z_{r,\varepsilon}|^{\lambda-\varepsilon})$, as claimed. □

Theorem XI.2.4

Proposition XI.2.14. Let f be an entire function of finite order λ. If $\varepsilon > 0$ then $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$ for all z with $|z|$ sufficiently large, and z can be found with $|z|$ as large as desired such that $|f(z)| \geq \exp(|z|^{\lambda+\varepsilon})$.

Proof. Since e^x is an increasing function, then for any nonzero $z \in \mathbb{C}$, if $a < b$ then $\exp(|z|^a) < \exp(|z|^b)$. For $\varepsilon > 0$, $\lambda + \varepsilon > \lambda$ so, since $\lambda = \inf\{a \mid |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\}$, we have for $|z|$ sufficiently large that $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$. Since $\lambda - \varepsilon < \lambda$ then $\lambda - \varepsilon \notin \{a \mid |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\}$. So for all $r > 0$ there is $z_{r,\varepsilon} \in \mathbb{C}$ with $|z_{r,\varepsilon}| > r$ and $|f(z_{r,,\varepsilon})| \geq \exp(|z_{r,\varepsilon}|^{\lambda-\varepsilon})$, as claimed. □