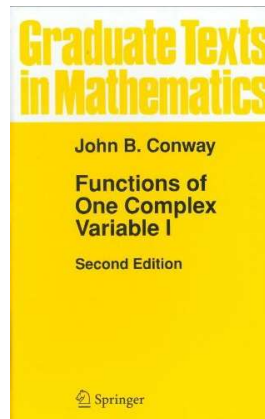


# Complex Analysis

## Chapter XII. The Range of an Analytic Function XII.2. The Little Picard Theorem—Proofs of Theorems



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Lemma XII.2.1

### Lemma XI.2.1 (continued 1)

**Proof (continued).** Then  $H$  is nonzero on  $G$  and so there is a branch of  $\log(H(z))$  defined on  $G$  by Theorem VIII.2.2 (the (a) implies (g) part), denoted  $g(z) = \log(H(z))$ . Hence for  $z \in G$  we have

$$\begin{aligned} \cosh(2g(z)) + 1 &= \frac{e^{2g(z)} + e^{-2g(z)}}{2} + 1 \\ &= \frac{1}{2} \left( \sqrt{F(z)} - \sqrt{F(z) - 1} + \frac{1}{\sqrt{F(z)} - \sqrt{F(z) - 1}} \right)^2 \\ &= \frac{1}{2} \left( \frac{(\sqrt{F(z)} - \sqrt{F(z) - 1})^2 + 1}{\sqrt{F(z)} - \sqrt{F(z) - 1}} \right)^2 \\ &= \frac{1}{2} \left( \frac{F(z) - 2\sqrt{F(z)}\sqrt{F(z) - 1} + F(z) - 1 + 1}{\sqrt{F(z)} - \sqrt{F(z) - 1}} \right)^2 \end{aligned}$$

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Lemma XII.2.1

### Lemma XI.2.1

**Lemma XII.2.1.** Let  $G$  be a simply connected region and suppose that  $f$  is an analytic function on  $G$  that does not assume the values 0 or 1. Then there is an analytic function  $g$  on  $G$  such that  $f(z) = -\exp(i\pi \cosh(2g(z)))$  for all  $z \in G$ .

**Proof.** Since by hypothesis  $f$  is nonzero on  $G$  and  $G$  is simply connected, then there is a branch  $\ell$  of  $\log(f(z))$  defined on  $F$  by Theorem VIII.2.2 (the (a) implies (g) part); that is,  $e^{\ell(z)} = f(z)$  for all  $z \in G$ . Let  $F(z) = (2\pi i)^{-1}\ell(z)$ . ASSUME  $F(z) = n$  for some  $n \in \mathbb{Z}$  and for some  $a \in G$ . Then  $f(z) = \exp(\ell(z)) = \exp(2\pi i F(z)) = \exp(2\pi i n) = 1$ , a CONTRADICTION to the hypothesis that  $f$  does not assume the value 1. So  $F$  does not assume any integer values. Since  $F$  is nonzero on  $G$  and  $G$  is simply connected, then a branch of  $\sqrt{F(z)}$  exists on  $G$  by Theorem VIII.2.2 (the (a) implies (h) part). Similarly, since  $F(z) \neq 1$  on  $G$  then a branch of  $\sqrt{F(z) - 1}$  exists on  $G$ . Let  $H(z) = \sqrt{F(z)} - \sqrt{F(z) - 1}$ .

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Lemma XII.2.1

### Lemma XI.2.1 (continued 2)

**Proof (continued).** ...

$$\begin{aligned} \cosh(2g(z)) + 1 &= \frac{1}{2} \left( \frac{2F(z) - 2\sqrt{F(z)}\sqrt{F(z) - 1}}{\sqrt{F(z)} - \sqrt{F(z) - 1}} \right)^2 \\ &= \frac{1}{2} \left( \frac{2F(z) - 2\sqrt{F(z)}\sqrt{F(z) - 1}}{\sqrt{F(z)} - \sqrt{F(z) - 1}} \right)^2 \\ &= \frac{1}{2} \left( \frac{2\sqrt{F(z)}(\sqrt{F(z)} - \sqrt{F(z) - 1})}{\sqrt{F(z)} - \sqrt{F(z) - 1}} \right)^2 \\ &= 2F(z) = \frac{2}{2\pi i} \ell(z), \end{aligned}$$

and so  $\ell(z) = \pi i(\cosh(2g(z)) + 1)$ . So  $f(z) = e^{\ell(z)} = \exp(\pi i + \pi i \cosh(2g(z))) = -\exp(\pi i \cosh(2g(z)))$ , as claimed. □

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**Lemma XII.2.2.** Let  $G$  be a simply connected region and suppose that  $f$  is an analytic function on  $G$  that does not assume the values 0 or 1. Let  $g$  be analytic on  $G$  where  $f(z) = -\exp(i\pi \cosh(2g(z)))$  for all  $z \in G$  (such  $g$  exists by Lemma XII.2.2). Then  $g(G)$  contains no disk of radius 1.

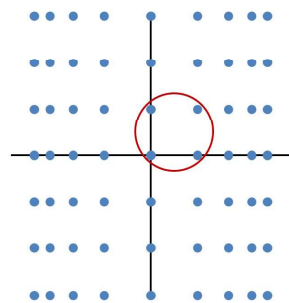
**Proof.** Let  $n, m \in \mathbb{Z}$  with  $n > 0$ . ASSUME there is  $z \in G$  with  $g(z) = \pm \log(\sqrt{n} + \sqrt{n-1}) + \frac{1}{2}im\pi$  (we use the principal branch of the log). Then

$$\begin{aligned} 2 \cosh(2g(a)) &= e^{2g(a)} + e^{-2g(a)} \\ &= \exp(\pm 2 \log(\sqrt{n} + \sqrt{n-1}) + im\pi) \\ &\quad + \exp(\mp 2 \log(\sqrt{n} + \sqrt{n-1}) + im\pi) \\ &= e^{im\pi} \left( (\sqrt{n} + \sqrt{n-1})^{\pm 2} + (\sqrt{n} + \sqrt{n-1})^{\mp 2} \right). \quad (*) \end{aligned}$$

Now

$$\begin{aligned} (\sqrt{n} + \sqrt{n-1})^{\pm 2} + (\sqrt{n} + \sqrt{n-1})^{\mp 2} &= (\sqrt{n} + \sqrt{n-1})^2 + (\sqrt{n} + \sqrt{n-1})^{-2} \\ &= (\sqrt{n} + \sqrt{n-1})^2 + \frac{1}{(\sqrt{n} + \sqrt{n-1})^2} \frac{(\sqrt{n} - \sqrt{n-1})^2}{(\sqrt{n} - \sqrt{n-1})^2} \end{aligned}$$

**Proof (continued).** The height for a rectangle with base on  $\text{Im}(z) = \frac{1}{2}m\pi$  is  $|\frac{1}{2}i(m+1)\pi - \frac{1}{2}im\pi| = \pi/2$  (and notice that  $\pi/2 < \sqrt{3}$ ). The width of a rectangle with left side on  $\text{Re}(z) = \log(\sqrt{n} + \sqrt{n-1})$  is  $\log(\sqrt{n+1} + \sqrt{n}) - \log(\sqrt{n} + \sqrt{n-1}) > 0$ . Now  $\varphi(x) = \log(\sqrt{x+1} + \sqrt{x}) - \log(\sqrt{x} + \sqrt{x-1})$  is a decreasing function for  $x \geq 1$  (check the first derivative) so that the width of such a rectangle is at most  $\varphi(1) = \log(\sqrt{1+1})$  (and notice that  $\log(\sqrt{2} + 1) < \log e = 1$ ). Similarly, the width of a rectangle with left side  $\text{Re}(z) = -\log(\sqrt{n} + \sqrt{n-1})$  is less than 1. So the diagonal of any such rectangle is less than  $\sqrt{(\sqrt{3})^2 + (1)^2} = 2$ . So  $g(G)$  excludes the grid of points in  $\mathbb{C}$  determining a set of rectangle of diameters less than 2. Hence  $g(G)$  cannot contain a disk of radius 1.  $\square$



**Proof (continued).**

$$\begin{aligned} &= (\sqrt{n} + \sqrt{n-1})^2 + \frac{(\sqrt{n} - \sqrt{n-1})^2}{(n - (n-1))^2} \\ &= (\sqrt{n} + \sqrt{n-1})^2 + (\sqrt{n} - \sqrt{n-1})^2 = 2(2n-1), \end{aligned}$$

so (\*) becomes  $2 \cosh(2g(a)) = (-1)^m 2(2n-1)$ , or  $\cosh(2g(a)) = (-1)^m (2n-1)$ . Therefore

$$f(a) = -\exp(i\pi \cosh(2g(a))) = -\exp(i\pi(-1)^m(2n-1)) = 1$$

since  $i\pi(-1)^m(2n-1)$  is an odd multiple of  $\pi i$  and so  $\exp(i\pi(-1)^m(2n-1)) = -1$ . But this is a CONTRADICTION to the hypothesis that  $f$  does not assume the value 1. So the assumption that there is  $a \in G$  such that  $g(a) = \pm \log(\sqrt{n} + \sqrt{n-1}) + \frac{1}{2}im\pi$  is false and hence  $g$  cannot assume any values in the set

$$\left\{ \pm \log(\sqrt{n} + \sqrt{n-1}) + \frac{1}{2}im\pi \mid \mathbb{Z}, n > 0 \right\}.$$

The points in this set form a grid in  $\mathbb{C}$  of rectangles of constant height and varying width.

## Theorem XII.2.3

### Theorem XII.2.3. The Little Picard Theorem.

If  $f$  is an entire function that omits two values then  $f$  is constant. That is, if  $f$  is a nonconstant entire function then it assumes every complex number with one possible exception.

**Proof.** Suppose that  $f$  omits  $a$  and  $b$  where  $a \neq b$ . With  $f(z) \neq a$  and  $f(z) \neq b$ , the entire function  $(f(z) - a)/(b - z)$  omits the values 0 and 1, so we can assume without loss of generality that  $f$  omits 0 and 1. By Lemma XII.2.2 there is an entire function  $g$  (entire since  $G = \mathbb{C}$  here) such that  $g(\mathbb{C})$  contains no disk of radius 1 and  $f(z) = -\exp(i\pi \cosh(2g(z)))$ . ASSUME  $f$  is not a constant function. Then  $g$  is not a constant function and so there is  $z_0 \in \mathbb{C}$  with  $g'(z_0) \neq 0$  by Proposition III.2.10. By considering  $g(z + z_0)$ , we can without loss of generality suppose that  $g'(0) \neq 0$ .

## Theorem XII.2.3 (continued)

**Theorem XII.2.3. The Little Picard Theorem.**

If  $f$  is an entire function that omits two values then  $f$  is constant. That is, if  $f$  is a nonconstant entire function then it assumes every complex number with one possible exception.

**Proof (continued).** But then by Corollary XII.1.11, for all  $R > 0$  (since  $g$  is entire),  $g(B(0; R))$  contains a disk of radius  $R/|g'(0)|L$  where  $L$  is Landau's constant (see Definition XII.1.9) and is approximately 1.2. But then, for  $R > 1/(|g'(0)|L)$  we have that  $g(\mathbb{C})$  contains a disk of radius greater than 1, CONTRADICTING Lemma XII.2.2. This contradiction shows that the assumption that  $f$  is not constant is false. So if  $f$  omits two values then  $f$  must be constant, as claimed.  $\square$