Complex Analysis

Chapter XII. The Range of an Analytic Function

XII.2. The Little Picard Theorem—Proofs of Theorems

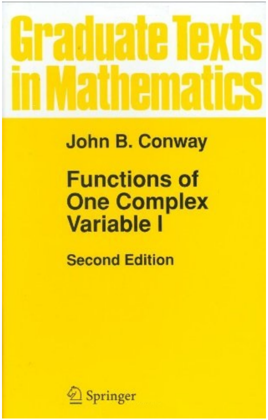

Table of contents

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Lemma XII.2.1. Let G be a simply connected region and suppose that f is an analytic function on G that does not assume the values 0 or 1. Then there is an analytic function g on G such that $f(z) = -\exp(i\pi\cosh(2g(z))$ for all $z \in G$.

Proof. Since by hypothesis f is nonzero on G and G is simply connected, then there is a branch ℓ of $\log(f(z))$ defined on F by Theorem VIII.2.2 (the (a) implies (g) part); that is, $e^{\ell(z)} = f(z)$ for all $z \in G$. Let $F(z) = (2\pi i)^{-1}\ell(z)$.

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Lemma XI.2.1 (continued 1)

Proof (continued). Then H is nonzero on G and so there is a branch of $\log(H(z))$ defined on G by Theorem VIII.2.2 (the (a) implies (g) part), denoted $g(z) = \log(H(z))$. Hence for $z \in G$ we have

$$
\begin{aligned}
\cosh(2g(z)) + 1 &= \frac{e^{2g(z)} + e^{-2g(z)}}{2} + 1 \\
&= \frac{1}{2}\left(\sqrt{F(z)} - \sqrt{F(z)-1} + \frac{1}{\sqrt{F(z)} - \sqrt{F(z)-1}}\right)^2 \\
&= \frac{1}{2}\left(\frac{(\sqrt{F(z)} - \sqrt{F(z)-1})^2 + 1}{\sqrt{F(z)} = \sqrt{F(z)-1}}\right)^2 \\
&= \frac{1}{2}\left(\frac{F(z) - 2\sqrt{F(z)}\sqrt{F(z)-1} + F(z) - 1 + 1}{\sqrt{F(z)} - \sqrt{F(z)-1}}\right)^2
\end{aligned}
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$$\begin{aligned}
\cosh(2g(z)) + 1 &= \frac{e^{2g(z)} + e^{-2g(z)}}{2} + 1 \\
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\end{aligned}$$

...

Lemma XI.2.1 (continued 2)

Proof (continued). ...

$$\begin{aligned}\cosh(2g(z))+1 &= \frac{1}{2}\left(\frac{2F(z)-2\sqrt{F(z)}\sqrt{F(z)-1}}{\sqrt{F(z)}-\sqrt{F(z)-1}}\right)^2 \\ &= \frac{1}{2}\left(\frac{2F(z)-2\sqrt{F(z)}\sqrt{F(z)-1}}{\sqrt{F(z)}-\sqrt{F(z)-1}}\right)^2 \\ &= \frac{1}{2}\left(\frac{2\sqrt{F(a)}(\sqrt{F(z)}=\sqrt{F(z)-1})}{\sqrt{F(z)}-\sqrt{F(z)-1}}\right)^2 \\ &= 2F(z)=\frac{2}{2\pi i}\ell(z),\end{aligned}$$

and so $\ell(z)=\pi i(\cosh(2g(z))+1)$. So $f(z)=e^{\ell(z)}=\exp(\pi i+\pi i\cosh(2g(z)))=-\exp(\pi i\cosh(2g(z)))$, as claimed. □

Lemma XII.2.2. Let G be a simply connected region and suppose that f is an analytic function on G that does not assume the values 0 or 1. Let g be analytic on G where $f(z) = -\exp(i\ pi \cosh(2g(z)))$ for all $z \in G$ (such g exists by Lemma XII.2.2). Then $g(G)$ contains no disk of radius 1.

Proof. Let $n, m \in \mathbb{Z}$ with $n > 0$. ASSUME there is $z \in G$ with $g(z) = \pm \log(\sqrt{n} + \sqrt{n-1}) + \frac{1}{2} im\pi$ (we use the principal branch of the log).

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$$\begin{aligned} 2\cosh(2g(a)) &= e^{2g(a)} + e^{-2g(a)} \\ &= \exp(\pm 2\log(\sqrt{n} + \sqrt{n-1}) + im\pi) \\ &\quad + \exp(\mp 2\log(\sqrt{n} + \sqrt{n-1}) + im\pi) \\ &= e^{im\pi}\left((\sqrt{n} + \sqrt{n-1})^{\pm 2} + (\sqrt{n} + \sqrt{n-1})^{\mp 2}\right). \quad (*) \end{aligned}$$

Lemma XII.2.2. Let G be a simply connected region and suppose that f is an analytic function on G that does not assume the values 0 or 1. Let g be analytic on G where $f(z) = -\exp(i\ pi \cosh(2g(z)))$ for all $z \in G$ (such g exists by Lemma XII.2.2). Then $g(G)$ contains no disk of radius 1.

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Now

$$(\sqrt{n}+\sqrt{n-1})^{\pm 2}+(\sqrt{n}+\sqrt{n-1})^{\mp 2} = (\sqrt{n}+\sqrt{n-1})^{2}+(\sqrt{n}+\sqrt{n-1})^{-2}$$

$$= (\sqrt{n}+\sqrt{n-1})^2 + \frac{1}{(\sqrt{n}+\sqrt{n-1})^2}\frac{(\sqrt{n}-\sqrt{n-1})^2}{(\sqrt{n}-\sqrt{n-1})^2}$$

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Proof (continued).

$$\begin{aligned} &= (\sqrt{n}+\sqrt{n-1})^2 + \frac{(\sqrt{n}-\sqrt{n-1})^2}{(n-(n-1))^2} \\ &= (\sqrt{n}+\sqrt{n-1})^2 + (\sqrt{n}-\sqrt{n-1})^2 = 2(2n-1), \end{aligned}$$

so $(*)$ becomes $2\cosh(2g(a)) = (-1)^m 2(2n-1)$, or $\cosh(2g(a)) = (-1)^m(2n-1)$. Therefore

$$f(a) = -\exp(i\pi\cosh(2g(a)) = -\exp(i\pi(-1)^m(2n-1)) = 1$$

since $i\pi(-1)^m(2n-1)$ is an odd multiple of πi and so $\exp(i\pi(-1)^m(2n-1) = -1$. But this is a CONTRADICTION to the hypothesis that f does not assume the value 1.

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$$\{\pm\log(\sqrt{n}+\sqrt{n-1}) + \frac{1}{2}im\pi \mid \mathbb{Z}, n > 0\}.$$

The points in this set form a grid in $\mathbb{C}$ of rectangles of constant height and varying width.

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Proof (continued). The height for a rectangle with base on $\text{Im}(z) = \frac{1}{2}m\pi$ is $|\frac{1}{2}i(m+1)\pi - \frac{1}{2}im\pi| = \pi/2$ (and notice that $\pi/2 < \sqrt{3}$). The width of a rectangle with left side on $\text{Re}(z) = \log(\sqrt{n} + \sqrt{n-1})$ is $\log(\sqrt{n+1} + \sqrt{n}) - \log(\sqrt{n} + \sqrt{n-1}) > 0$. Now $\varphi(x) = \log(\sqrt{x+1} + \sqrt{x}) - \log(\sqrt{x} - \sqrt{x-1})$ is a decreasing function for $x \geq 1$ (check the first derivative) so that the width of such a rectangle is at most $\varphi(1) = \log(\sqrt{1} + 1)$ (and notice that $\log(\sqrt{2} + 1) < \log e = 1$). Similarly, the width of a rectangle with left side $\text{Re}(z) = -\log(\sqrt{n} + \sqrt{n-1})$ is less than 1.

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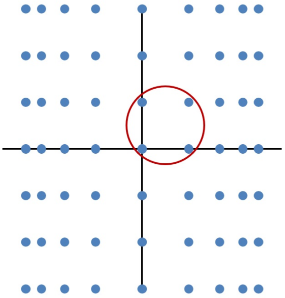
So the diagonal of any such rectangle is less than $\sqrt{(\sqrt{3})^2 + (1)^2} = 2$. So $g(G)$ excludes the grid of points in $\mathbb{C}$ determining a set of rectangle of diameters less than 2. Hence $g(G)$ cannot contain a disk of radius 1. □



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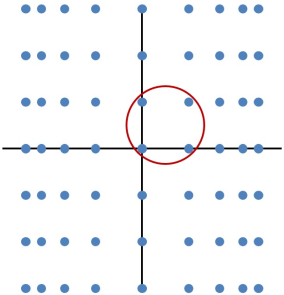

Theorem XII.2.3

Theorem XII.2.3. The Little Picard Theorem.
If f is an entire function that omits two values then f is constant. That is, if f is a nonconstant entire function then it assumes every complex number with one possible exception.

Proof. Suppose that f omits a and b where $a \neq b$. With $f(z) \neq a$ and $f(z) \neq b$, the entire function $(f(z) - a)/(b - z)$ omits the values 0 and 1, so we can assume without loss of generality that f omits 0 and 1. By Lemma XII.2.2 there is an entire function g (entire since $G = \mathbb{C}$ here) such that $g(\mathbb{C})$ contains no disk of radius 1 and $f(z) = -\exp(i\pi \cosh(2g(z)))$.

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Proof (continued). But then by Corollary XII.1.11, for all $R > 0$ (since g is entire), $g(B(0;R))$ contains a disk of radius $R|g'(0)|L$ where L is Landau's constant (see Definition XII.1.9) and is approximately 1.2. But then, for $R > 1/(|g'(0)|L)$ we have that $g(\mathbb{C})$ contains a disk of radius greater than 1, CONTRADICTING Lemma XII.2.2.

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