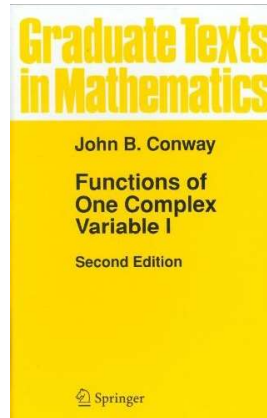


Complex Analysis

Chapter XII. The Range of an Analytic Function XII.3. Schottky's Theorem—Proofs of Theorems



Theorem XII.3.1

Theorem XII.3.1. Schottky's Theorem.

For each α and β with $0 < \alpha < \infty$ and $0 \leq \beta < 1$, there is a constant $C(\alpha, \beta)$ such that f is an analytic function on some simply connected region containing $\overline{B}(0; 1)$ that omits the values 0 and 1, and such that $|f(0)| \leq \alpha$, where $|f(z)| \leq C(\alpha, \beta)$ for $|z| \leq \beta$.

Proof. We give a proof for $2 \leq \alpha < \infty$. Since the hypothesis $|f(0)| \leq \alpha$ where $0 < \alpha \leq 2$ satisfies $|f(0)| \leq \alpha \leq 2$, this approach covers all values of α .

Case 1. Suppose $1/2 \leq |f(0)| \leq \alpha$. Then, with F, H, g , and ℓ as discussed above,

$$\begin{aligned} |F(0)| &= \left| \frac{1}{2\pi i} \ell(0) \right| = \left| \frac{1}{2\pi i} \log(f(0)) \right| = \frac{1}{2\pi} |\log |f(0)| + i \operatorname{Im}(\ell(0))| \\ &\leq \frac{1}{2\pi} (|\log |f(0)|| + \operatorname{Im}(f(0))) \leq \frac{1}{2\pi} (\log(\alpha) + 2\pi) = \frac{1}{2\pi} \log(\alpha) + 1. \end{aligned}$$

Let $C_0(\alpha) = \frac{1}{2\pi} \log(\alpha) + 1$.

Theorem XII.3.1 (continued 1)

Proof (continued). So $|F(0)| \leq C_0(\alpha)$ and $|F(0) - 1| \leq |F(0)| + 1 \leq C_0(\alpha) + 1$. Also

$$\begin{aligned} |\sqrt{F(0)} \pm \sqrt{F(0) - 1}| &\leq |\sqrt{F(z)}| + |\sqrt{F(0) - 1}| \\ &= \exp\left(\frac{1}{2} \log |F(0)|\right) + \exp\left(\frac{1}{2} \log |F(0) - 1|\right) \\ &= |F(0)|^{1/2} + |F(0) - 1|^{1/2} \leq C_0(\alpha)^{1/2} + (C_0(\alpha) + 1)^{1/2}. \end{aligned}$$

Let $C_1(\alpha) = C_0(\alpha)^{1/2} + (C_0(\alpha) + 1)^{1/2}$. Now if $|H(0)| = |\sqrt{F(0)} - \sqrt{F(0) - 1}| \geq 1$ then $\log |H(0)| \geq 0$ and so

$$\begin{aligned} |g(0)| &= |\log |H(0)|| = |\log |H(0)| + i \operatorname{Im}(g(0))| \\ &\leq \log |H(0)| + \operatorname{Im}(g(0)) \leq \log |H(0)| + 2\pi \\ &\leq \log |\sqrt{F(0)} - \sqrt{F(0) - 1}| + 2\pi \leq \log(C_1(\alpha)) + 2\pi. \end{aligned}$$

Theorem XII.3.1 (continued 2)

Proof (continued). If $|H(0)| = |\sqrt{F(0)} - \sqrt{F(0) - 1}| < 1$ then $\log |H(0)| < 0$ and so

$$\begin{aligned} |g(0)| &= |\log |H(0)|| = |\log |H(0)| + i \operatorname{Im}(g(0))| \\ &\leq \log |H(0)| + \operatorname{Im}(g(0)) = -\log |H(0)| + \operatorname{Im}(g(0)) \\ &\leq -\log |H(0)| + 2\pi = \log \frac{1}{|H(0)|} + 2\pi \\ &= \log \frac{1}{\sqrt{F(0)} - \sqrt{F(0) - 1}} + 2\pi \\ &= \log \frac{1}{\sqrt{F(0)} - \sqrt{F(0) - 1}} \frac{\sqrt{F(z)} + \sqrt{F(z) - 1}}{\sqrt{F(z)} + \sqrt{F(z) - 1}} + 2\pi \\ &= \log |\sqrt{F(0)} + \sqrt{F(0) - 1}| + 2\pi \leq \log(C_1(\alpha)) + 2\pi. \end{aligned}$$

Let $C_2(\alpha) = \log(C_1(\alpha)) + 2\pi$.

Theorem XII.3.1 (continued 3)

Proof (continued). Let $a \in \mathbb{C}$ with $|a| < 1$. Then Corollary XII.1.11 implies that $g(B(a; 1 - |a|))$ contains a disk of radius $L|g'(a)|(1 - |a|)$ where L is Landau's constant (it is approximately $1/2$; see Definition XII.1.9). By Lemma XII.2.2, $g(B(0; 1))$ contains no disk of radius 1, so it must be that $L|g'(a)|(1 - |a|) < 1$; that is

$$|g'(a)| < \frac{1}{L(1 - |a|)} \text{ for } |a| < 1. \quad (3.6)$$

If $a \in \mathbb{C}$ with $|a| < 1$, then let γ be the line segment $[0, a]$. Then

$$\begin{aligned} |g(a)| &\leq |g(a) + g(0) - g(0)| \leq |g(0)| + |g(a) - g(0)| \\ &\leq C_2(\alpha) + \left| \int_{\gamma} g'(z) dz \right| \leq C_2(\alpha) + |a| \max_{z \in [0, a]} |g'(z)|. \end{aligned}$$

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Theorem XII.3.1 (continued 5)

Proof (continued). Consequently, if $|z| \leq \beta$ then

$$\begin{aligned} |f(z)| &= \exp(\pi i \cosh(2g(z))) \text{ by Lemma XII.2.1} \\ &\leq \exp(\pi |\cosh(2g(z))|) \\ &\leq \exp(\pi e^{2|g(z)|}) \text{ since } |\cosh(2g(z))| = |e^{2g(z)} + e^{-2g(z)}|/2 \\ &\leq (|e^{2g(z)}| + |e^{-2g(z)}|)/2 \leq (2e^{2|g(z)|})/2 = e^{2|g(z)|} \\ &\leq \exp(\pi e^{2C_3(\alpha, \beta)}). \end{aligned}$$

Define $C_4(\alpha, \beta) = \exp(\pi e^{2C_3(\alpha, \beta)})$. Then for $1/2 \leq |f(0)| \leq \alpha$ and $|z| \leq \beta$ we now have $|f(z)| \leq C_4(\alpha, \beta)$.

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Theorem XII.3.1 (continued 4)

Proof (continued). Now (3.6) holds for all $a \in \mathbb{C}$ with $|a| < 1$, so for each $z \in [0, a]$ we have

$$\begin{aligned} |g'(z)| &< \frac{1}{L(1 - |z|)} \text{ replacing } |a| < 1 \text{ with } |z| \leq |a| < 1 \\ &\leq \frac{1}{L(1 - |a|)} \text{ since } |z| \leq |a| \text{ here.} \end{aligned}$$

So $|g(a)| \leq C_2(\alpha) + \frac{|a|}{L(1 - |a|)}$. That is (replacing $|a| < 1$ with $|z| < 1$), for $|z| < 1$

$$|g(z)| \leq C_2(\alpha) + \frac{|z|}{L(1 - |z|)} \leq C_2(\alpha) + \frac{\beta}{L(1 - \beta)}$$

for $|z| \leq \beta < 1$ (the last inequality holds since $f(x) = x/(1 - x)$ is an increasing function for $x \neq 0$). With $C_3(\alpha, \beta) = C_2(\alpha) + \frac{\beta}{L(1 - \beta)}$ we have $|g(z)| \leq C_3(\alpha, \beta)$ if $|z| \leq \beta$.

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Theorem XII.3.1 (continued 6)

Theorem XII.3.1. Schottky's Theorem.

For each α and β with $0 < \alpha < \infty$ and $0 \leq \beta < 1$, there is a constant $C(\alpha, \beta)$ such that f is an analytic function on some simply connected region containing $\overline{B}(0; 1)$ that omits the values 0 and 1, and such that $|f(0)| \leq \alpha$, where $|f(z)| \leq C(\alpha, \beta)$ for $|z| \leq \beta$.

Proof (continued). Case 2. Suppose $0 < |f(0)| \leq 1/2$. Then the function $1 - f$ satisfies the conditions of Case 1 (namely, $1/2 \leq |1 - f(0)| \leq 1 \leq \alpha$ with $\alpha = 2$). So by Case 1, $|1 - f(z)| \leq C_4(2, \beta)$ for $|z| \leq \beta$. Hence $|f(z)| - 1 \leq |1 - f(z)| \leq C_4(2, \beta)$ and $|f(z)| \leq C_4(2, \beta) + 1$ for $0 < |f(0)| < 1/2$ and $|z| \leq \beta$.

Therefore, with $C(\alpha, \beta) = \max\{C_4(\alpha, \beta), C_4(\alpha, \beta) + 1\}$ we have $|f(z)| \leq C(\alpha, \beta)$ for all $0 < \alpha < \infty$ and $0 \leq \beta < 1$ where $|f(0)| \leq \alpha$, as claimed. \square

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