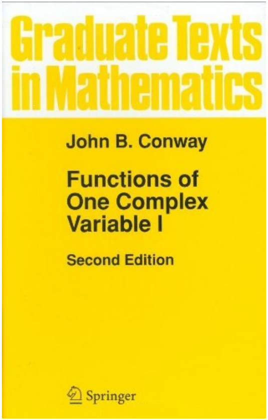
Complex Analysis

Chapter XII. The Range of an Analytic Function

XII.3. Schottky's Theorem—Proofs of Theorems



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Case 1. Suppose $1/2 \leq |f(0)| \leq \alpha$. Then, with F, H, g, and ℓ as discussed above,

$$|F(0)| = \left|\frac{1}{2\pi i}\ell(0)\right| = \left|\frac{1}{2\pi i}\log(f(0))\right| = \frac{1}{2\pi}\left|\log|f(0)) + i\text{Im}(\ell(0))\right|$$

$$\leq \frac{1}{2\pi}(|\log|f(0)|| + \text{Im}(f(0)) \leq \frac{1}{2\pi}(\log(\alpha) + 2\pi) = \frac{1}{2\pi}\log(\alpha) + 1.$$

Let $C_0(\alpha) = \frac{1}{2\pi}\log(\alpha) + 1$.

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Let $C_0(\alpha) = \frac{1}{2\pi}\log(\alpha) + 1$.

Theorem XII.3.1 (continued 1)

Proof (continued). So $|F(0)| \le C_0(\alpha)$ and $|F(0)-1| \le |F(0)|+1 \le C_0(\alpha)+1$. Also

$$|\sqrt{F(0)} \pm \sqrt{F(0)-1}| \le |\sqrt{F(z)}| + |\sqrt{F(0)-1}|$$

$$= \exp\left(\frac{1}{2}\log|F(0)|\right) + \exp\left(\frac{1}{2}\log|F(0)-1|\right)$$

$$= |F(0)|^{1/2} + |F(0)-1|^{1/2} \le C_0(\alpha)^{1/2} + (C_0(\alpha)+1)^{1/2}.$$

Let $C_1(\alpha) = C_0(\alpha)^{1/2} + (C_0(\alpha)+1)^{1/2}$. Now if $|H(0)| = |\sqrt{F(0)} - \sqrt{F(0)-1}| \ge 1$ then $\log|H(0)| \ge 0$ and so

$$\begin{aligned} |g(0)| &= |\log|H(0)| = |\log|H(0)| + i\text{Im}(g(0))| \\ &\le \log|H(0)| + \text{Im}(g(0)) \le \log|H(0)| + 2\pi \\ &\le \log|\sqrt{F(0)}| - \sqrt{F(0)-1}| + 2\pi \le \log(C_1(\alpha)) + 2\pi. \end{aligned}$$

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$$\begin{aligned} |g(0)| &= |\log|H(0)| = |\log|H(0)| + i\mathrm{Im}(g(0))| \\ &\le \log|H(0)| + \mathrm{Im}(g(0)) \le \log|H(0)| + 2\pi \\ &\le \log|\sqrt{F(0)}| - \sqrt{F(0)-1}| + 2\pi \le \log(C_1(\alpha)) + 2\pi. \end{aligned}$$

Theorem XII.3.1 (continued 2)

Proof (continued). If $|H(0)| = |\sqrt{F(0)} - \sqrt{F(0) - 1}| < 1$ then $\log|H(0)| < 0$ and so

$$\begin{aligned}
|g(0)| &= |\log(H(0))| = |\log|H(0)| + i\mathrm{Im}(g(0))| \\
&\leq \log|H(0)| + \mathrm{Im}(g(0)) = -\log|H(0| + \mathrm{Im}(g(0)) \\
&\leq -\log|H(0)| + 2\pi = \log\frac{1}{|H(0)|} + 2\pi \\
&= \log\frac{1}{\sqrt{F(0)} - \sqrt{F(0) - 1}} + 2\pi \\
&= \log\frac{1}{\sqrt{F(0)} - \sqrt{F(0) - 1}}\frac{\sqrt{F(z)} + \sqrt{F(z) - 1}}{\sqrt{F(z)} + \sqrt{F(z) - 1}} + 2\pi \\
&= \log|\sqrt{F(0} + \sqrt{F(0) - 1}| + 2\pi \leq \log(C_1(\alpha)) + 2\pi.
\end{aligned}$$

Let $C_2(\alpha) = \log(C_1(\alpha)) + 2\pi$.

Theorem XII.3.1 (continued 3)

Proof (continued). Let $a \in \mathbb{C}$ with $|a| < 1$. Then Corollary XII.1.11 implies that $g(B(a; 1 - |a|)$ contains a disk of radius $L|g'(a)|(1 - |a|)$ where L is Landau's constant (it is approximately $1/2$; see Definition XII.1.9). By Lemma XII.2.2, $g(B(0; 1))$ contains no disk of radius 1, so it must be that $L|g'(a)|(1 - |a|) < 1$; that is

$$|g'(a)| < \frac{1}{L(1 - |a|)} \text{ for } |a| < 1. \quad (3.6)$$

If $a \in \mathbb{C}$ with $|a| < 1$, then let γ be the line segment $[0, a]$. Then

$$\begin{aligned} |g(a)| &\leq |g(a) + g(0) - g(0)| \leq |g(0)| + |g(a) - g(0)| \\ &\leq C_2(\alpha) + \left| \int_\gamma g'(z)\, dz \right| \leq C_2(\alpha) + |a| \max_{z \in [0,a]} |g'(z)|. \end{aligned}$$

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Theorem XII.3.1 (continued 4)

Proof (continued). Now (3.6) holds for all $a \in \mathbb{C}$ with $|a| < 1$, so for each $z \in [0, a]$ we have

$$\begin{aligned} |g'(z)| &< \frac{1}{L(1-|z|)} \text{ replacing } |a| < 1 \text{ with } |z| \le |a| < 1 \\ &\le \frac{1}{L(1-|a|)} \text{ since } |z| \le |z| \text{ here.} \end{aligned}$$

So $|g(a)| \le C_2(\alpha) + \frac{|a|}{L(1-|a|)}$. That is (replacing $|a| < 1$ with $|z| < 1$), for $|z| < 1$

$$|g(z)| \le C_2(\alpha) + \frac{|z|}{L(1-|z|)} \le C_2(\alpha) + \frac{\beta}{L(1-\beta)}$$

for $|z| \le \beta < 1$ (the last inequality holds since $f(x) = x/(1-x)$ is an increasing function for $x \neq 0$). With $C_3(\alpha, \beta) = C_2(\alpha) + \frac{\beta}{L(1-\beta)}$ we have $|g(z)| \le C_3(\alpha, \beta)$ if $|z| \le \beta$.

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Theorem XII.3.1 (continued 5)

Proof (continued). Consequently, if $|z| \leq \beta$ then

$$\begin{aligned} |f(z)| &= \exp(\pi i \cosh(2g(z))| \text{ by Lemma XII.2.1} \\ &\leq \exp(\pi|\cosh(2g(z))|) \\ &\leq \exp(\pi e^{2|g(z)|}) \text{ since } |\cosh(2g(z))| = |e^{2g(z)} + e^{-2g(z)}|/2 \\ &\quad \leq (|e^{2g(z)}| + |e^{-2g(z)}|)/2 \leq (2e^{2|g(z)|})/2 = e^{2|g(z)|} \\ &\leq \exp(\pi e^{2C_3(\alpha,\beta)}). \end{aligned}$$

Define $C_4(\alpha, \beta) = \exp(\pi e^{2C_3(\alpha,\beta)})$. Then for $1/2 \leq |f(0)| \leq \alpha$ and $|z| \leq \beta$ we now have $|f(z)| \leq C_4(\alpha, \beta)$.

Theorem XII.3.1 (continued 6)

Theorem XII.3.1. Schottky's Theorem.
For each α and β with $0 < \alpha < \infty$ and $0 \leq \beta < 1$, there is a constant $C(\alpha, \beta)$ such that f is an analytic function on some simply connected region containing $\overline{B}(0;1)$ that omits the values 0 and 1, and such that $|f(0)| \leq \alpha$, where $|f(z)| \leq C(\alpha, \beta)$ for $|z| \leq \beta$.

Proof (continued). Case 2. Suppose $0 < |f(0)| \leq 1/2$. Then the function $1 - f$ satisfies the conditions of Case 1 (namely, $1/2 \leq |1 - f(0)| \leq 1 \leq \alpha$ with $\alpha = 2$). So by Case 1, $|1 - f(z)| \leq C_4(2, \beta)$ for $|z| \leq \beta$. Hence $|f(z)| - 1 \leq |1 - f(z)| \leq C_4(2, \beta)$ and $|f(z)| \leq C_4(2, \beta) + 1$ for $0 < |f(0)| < 1/2$ and $|z| \leq \beta$.

Therefore, with $C(\alpha, \beta) = \max\{C_4(\alpha, \beta), C_4(\alpha, \beta) + 1\}$ we have $|f(z)| \leq C(\alpha, \beta)$ for all $0 < \alpha < \infty$ and $0 \leq \beta < 1$ where $|f(0)| \leq \alpha$, as claimed. □

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Therefore, with $C(\alpha, \beta) = \max\{C_4(\alpha, \beta), C_4(\alpha, \beta) + 1\}$ we have $|f(z)| \leq C(\alpha, \beta)$ for all $0 < \alpha < \infty$ and $0 \leq \beta < 1$ where $|f(0)| \leq \alpha$, as claimed. □