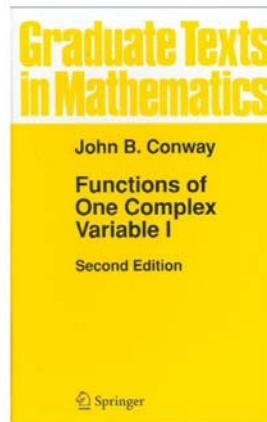


Complex Analysis

Chapter XII. The Range of an Analytic Function XII.4. The Great Picard Theorem—Proofs of Theorems



Theorem XII.4.1

Theorem XII.4.1. Montel-Carathéodory Theorem.

If \mathcal{F} is the family of all analytic functions on a region G that do not assume the values 0 and 1, then \mathcal{F} is normal in $C(G, \mathbb{C}_\infty)$.

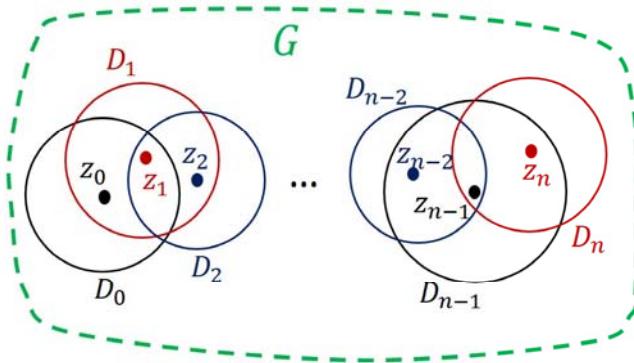
Proof. Fix a point $z_0 \in G$ and define

$$\mathcal{G} = \{f \in \mathcal{F} \mid |f(z_0)| \leq 1\}, \quad \mathcal{H} = \{f \in \mathcal{F} \mid |f(z_0)| \geq 1\}.$$

So $\mathcal{F} = \mathcal{G} \cup \mathcal{H}$. We will show that \mathcal{G} is normal in $H(G)$ (where $H(G)$ is the collection of functions analytic on G) and that \mathcal{H} is normal in $C(G, \mathbb{C}_\infty)$ (notice that the function which is a constant of ∞ is a limit of a sequence of certain constant functions in \mathcal{H}). Recall that a set is normal in a function space if every sequence in the set has a subsequence convergent to an element of the space; by Montel's Theorem (Theorem VII.2.9), \mathcal{G} is normal in $H(G)$ if and only if \mathcal{G} is locally bounded (that is, for each point $a \in G$ there are constants M and $r > 0$ such that for all $f \in \mathcal{G}$ we have $|f(z)| \leq M$ for $|z - a| < r$).

Theorem XII.4.1 (continued 1)

Proof (continued). Let a be any point in G and let γ be a curve in G from z_0 to a . Let D_0, D_1, \dots, D_n be disks in G with centers $z_0, z_1, \dots, z_n = a$ on $\{\gamma\}$ and such that z_{k-1} and z_k are in $D_{k-1} \cap D_k$ for $1 \leq k \leq n$. Also assume $D_k^- \subset G$ for $0 \leq k \leq n$.



Theorem XII.4.1 (continued 2)

Proof (continued). We apply Schottky's Theorem (Theorem XII.3.3) to D_0 . If $D_0 = B(z_0; r)$ and $R > r$ is such that $\overline{B}(z_0; R) \subset G$ then, by Corollary XII.3.7, there is a constant $C(1, \beta)$ (we have $\alpha = 1$ since $|f(z_0)| \leq 1 = \alpha$ because $f \in \mathcal{G}$) such that $|f(z)| \leq C(1, \beta)$ for $z \in D_0$ provided β is chosen so that $r < \beta R$ and this bound holds for all $f \in \mathcal{G}$ by Schottky's Theorem. (Actually, we need to replace z with $z - z_0$ in Corollary XII.3.7 so that $f(z - z_0)|_{z=z_0} = f(0)$ and " $|f(z)| \leq C(\alpha, \beta)$ for $|z| \leq \beta R$ " becomes " $|f(z)| \leq C(\alpha, \beta)$ for $z \in \overline{B}(z_0; \beta R)$ "; then $|f(z)| \leq C(\alpha, \beta)$ for $z \in D_0 = B(z_0; r) \subset \overline{B}(z_0; \beta R)$ since $r < \beta R$.) With $C_0 = C(1, \beta)$ we have $|f(z)| \leq C_0$ for all $z \in D_0$ and for all $f \in \mathcal{G}$. Since $z_1 \in D_0$ then $|f(z_1)| \leq C_0$. Since this holds for all $f \in \mathcal{G}$, then by Schottky's Theorem and Corollary XII.3.7 there is C_1 where $|f(z)| \leq C_1$ for all $z \in D_1$ and for all $f \in \mathcal{G}$. Similarly, by induction, there is C_n such that $|f(z)| \leq C_n$ for all $z \in D_n$ and for all $f \in \mathcal{G}$. Since a is an arbitrary element of region G , then \mathcal{G} is locally bounded on G . So by Montel's Theorem (Theorem VII.2.9) \mathcal{G} is normal in $H(G) \subset C(G, \mathbb{C}_\infty)$.

Theorem XII.4.1 (continued 3)

Proof (continued). Now consider $\mathcal{H} = \{f \in \mathcal{F} \mid |f(z_0)| \geq 1\}$. For $f \in \mathcal{H} \subset \mathcal{F}$, a/f is analytic since f does not assume the value 0. Since f never assumes the value 1, then neither does $1/f$. Moreover, $|(1/f)(z_0)| = |1/f(z_0)| \leq 1$. So $\tilde{\mathcal{H}} = \{1/f \mid f \in \mathcal{H}\} \subset \mathcal{G}$ and so, by the above argument, $\tilde{\mathcal{H}}$ is normal in $H(G)$. So, by the definition of “normal,” if $\{f_n\}$ is a sequence in \mathcal{H} there is a subsequence $\{f_{n_k}\}$ and an analytic function h on G (i.e., $h \in H(G)$) such that $\{1/f_{n_k}\}$ converges in $H(G)$ to h . By Corollary VII.2.6 (a corollary to Hurwitz’s Theorem, Theorem VII.2.5) either $h \equiv 0$ or h is never 0 on G . If $h \equiv 0$ then “it is easy to see that” (we leave the details to Exercise XII.4.A) $f_{n_k} \rightarrow \infty$ uniformly on compact subsets of G in $C(G, \mathbb{C}_\infty)$. If h never vanishes then $1/h$ is analytic and $f_{n_k}(z) \rightarrow 1/h(z)$ uniformly on compact subsets of G . So by Proposition VII.1.10(b), $f_{n_k} \rightarrow 1/h$ in $C(G, \mathbb{C}_\infty)$. In either case, sequence $\{f_n\}$ in \mathcal{H} has a subsequence $\{f_{n_k}\}$ which converges in $C(G, \mathbb{C}_\infty)$; that is, by definition, \mathcal{H} is normal in $C(G, \mathbb{C}_\infty)$. So the claim holds. \square

Theorem XII.4.2

Theorem XII.4.2. The Great Picard Theorem.

Suppose an analytic function f has an essential singularity at $z = a$. Then in each neighborhood of a , f assumes each complex number, with one possible exception, an infinite number of times.

Proof. Without loss of generality, we take $a = 0$. ASSUME there is $R > 0$ such that there are two numbers not in $\{f(z) \mid 0 < |z| < R\}$. Without loss of generality we assume the two omitted values are 0 and 1 for $0 < |z| < R$ (otherwise, we can replace $f(z)$ with $(f(z) - a)/(b - a)$ where a and b are the omitted values). Let $G = B(0; R) \setminus \{0\}$ and define $f_n : G \rightarrow \mathbb{C}$ by $f_n(z) = f(z/n)$. So each f_n is analytic on G and no f_n assumes the value 0 or 1. By the Montel-Carathéodory Theorem (Theorem XII.4.1), $\{f_n\}$ is a normal family in $C(G, \mathbb{C}_\infty)$.

Theorem XII.4.2 (continued 1)

Proof (continued). Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that $f_{n_k} \rightarrow \varphi$ in $C(G, \mathbb{C}_\infty)$. Then by Proposition VII.1.10(b), $f_{n_k} \rightarrow \varphi$ uniformly on compact subsets of G and so $f_{n_k} \rightarrow \varphi$ uniformly on $\{z \mid |z| = R/2\}$. Since each f_{n_k} is analytic on G and the convergence is in $C(G, \mathbb{C}_\infty)$, then either φ is analytic or $\varphi \equiv \infty$ (the uniform convergence on $|z| = R/2$ implies continuity of φ so if $\varphi = \infty$ at any point then it must be ∞ throughout $G = B(0; R) \setminus \{0\}$). ASSUME φ is analytic. Let $M = \max\{|\varphi(z)| \mid |z| = R/2\}$. Then $|f(z/n_k)| = |f_{n_k}(z)| = |\varphi(z) + \varphi(z)| \leq |f_{n_k}(z) - \varphi(z)| + |\varphi(z)| \leq 2M$ for n_k sufficiently large and $|z| = R/2$. Thus $|f(z)| \leq 2M$ for $|z| = R/(2n_k)$ and for sufficiently large n_k (say $k \geq K$). So $|f(z)| \leq 2M$ on $\text{ann}(0; R/(2n_{k_1}), R/(2n_{k_2}))^-$ for all $k_1 > k_2 > K$ by the Maximum Modulus Theorem—2nd Version (Theorem VI.1.2). Since $n_k \rightarrow \infty$ then f is bounded by $2M$ on a deleted neighborhood of zero and so by Theorem V.1.2, f has a removable singularity at $z = 0$, CONTRADICTING the fact that f has an essential singularity at $z = 0$. So $\varphi \equiv \infty$ on G .

Theorem XII.4.2 (continued 2)

Theorem XII.4.2. The Great Picard Theorem.

Suppose an analytic function f has an essential singularity at $z = a$. Then in each neighborhood of a , f assumes each complex number, with one possible exception, an infinite number of times.

Proof (continued). But then, by Exercise XII.4.B, f must have a pole at $z = 0$, again a CONTRADICTION. So the original assumption that f omits two values is false and f cannot omit two values.

To show that f assumes every value an infinite number of times (with one possible exception). Suppose some complex w is assumed in $G \subset B(0; R) \setminus \{0\}$ only finitely many times (so w is not the possibly exceptional point). Then we can find a z value of smallest modulus at which f takes on the value w , and then repeat the above argument to show that there is a punctured disk in which f fails to assume two values which yields a contradiction. \square

Corollary XII.4.4

Corollary XII.4.4. If f is an entire function that is not a polynomial then f assumes every complex number, with one possible exception, an infinite number of times.

Proof. Consider $g(z) = f(1/z)$. Since f is not a polynomial then the power series for f contains infinitely many nonzero coefficients and so g has an essential singularity at $z = 0$. By the Great Picard Theorem (Theorem XII.4.2), g assumes every complex number with one possible exception. So f has the same property (if $w = g(z)$ where $z \neq 0$ then $f(1/z) = w$). \square