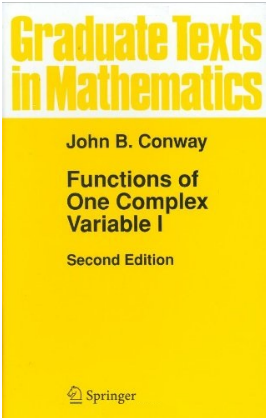
Complex Analysis

Chapter XII. The Range of an Analytic Function

XII.4. The Great Picard Theorem—Proofs of Theorems



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Proof. Fix a point $z_0 \in G$ and define

$$\mathcal{G} = \{f \in \mathcal{F} \mid |f(z_0)| \leq 1\},\ \mathcal{H} = \{f \in \mathcal{F} \mid |f(z_0)| \geq 1\}.$$

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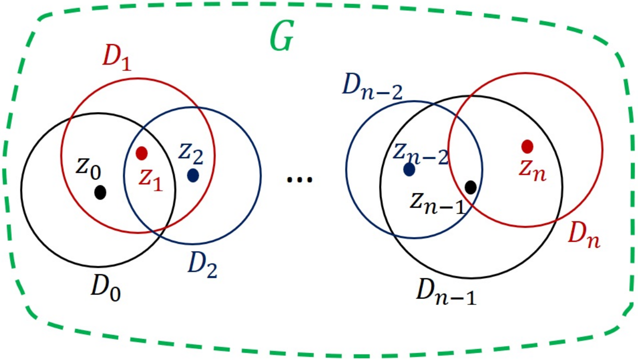
Theorem XII.4.1 (continued 1)

Proof (continued). Let a be any point in G and let γ be a curve in G from z_0 to a. Let $D_0, D_1, \ldots, D_n$ be disks in G with centers $z_0, z_1, \ldots, z_n = a$ on $\{\gamma\}$ and such that z_{k-1} and z_k are in $D_{k-1} \cap D_k$ for $1 \le k \le n$. Also assume $D_k^- \subset G$ for $0 \le k \le n$.



Theorem XII.4.1 (continued 2)

Proof (continued). We apply Schottky's Theorem (Theorem XII.3.3) to D_0. If $D_0 = B(z_0; r)$ and $R > r$ is such that $\overline{B}(z_0; R) \subset G$ then, by Corollary XII.3.7, there is a constant $C(1, \beta)$ (we have $\alpha = 1$ since $|f(z_0)| \leq 1 = \alpha$ because $f \in \mathcal{G}$) such that $|f(z)| \leq C(1, \beta)$ for $z \in D_0$ provided β is chosen so that $r < \beta R$ and this bound holds for all $f \in \mathcal{G}$ by Schottky's Theorem. (Actually, we need to replace z with $z - z_0$ in Corollary XII.3.7 so that $f(z - z_0)|_{z=z_0} = f(0)$ and "$|f(z)| \leq C(\alpha, \beta)$ for $|z| \leq \beta R$" becomes "$|f(z)| \leq C(\alpha, \beta)$ for $z \in \overline{B}(z_0; \beta R)$"; then $|f(z)| \leq C(\alpha, \beta)$ for $z \in D_0 = B(z_0; r) \subset \overline{B}(z_0; \beta R)$ since $r < \beta R$.) With $C_0 = C(1\beta)$ we have $|f(z)| \leq C_0$ for all $z \in D_0$ and for all $f \in \mathcal{G}$. Since $x_1 \in D_0$ then $|f(z_1)| \leq C_0$. Since this holds for all $f \in \mathcal{G}$, then by Schottky's Theorem and Corollary XII.3.7 there is C_1 where $|f(z)| \leq C_1$ for all $z \in D_1$ and for all $f \in \mathcal{G}$.

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Theorem XII.4.1 (continued 3)

Proof (continued). Now consider $\mathcal{H} = \{f \in \mathcal{F} \mid |f(z_0)| \geq 1\}$. For $f \in \mathcal{H} \subset \mathcal{F}$, a/f is analytic since f does not assume the value 0. Since f never assumes the value 1, then neither does $1/f$. Moreover, $|(1/f)(z_0)| = |1/f(z_0)| \leq 1$. So $\tilde{\mathcal{H}} = \{1/f \mid f \in \mathcal{H}\} \subset \mathcal{G}$ and so, by the above argument, $\tilde{\mathcal{H}}$ is normal in $H(G)$. So, by the definition of "normal," if $\{f_n\}$ is a sequence in $\mathcal{H}$ there is a subsequence $\{f_{n_k}\}$ and an analytic function h on G (i.e., $h \in H(G)$) such that $\{1/f_{n_k}\}$ converges in $H(G)$ to h. By Corollary VII.2.6 (a corollary to Hurwitz's Theorem, Theorem VII.2.5) either $h \equiv 0$ or h is never 0 on G.

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Theorem XII.4.2. The Great Picard Theorem.
Suppose an analytic function f has an essential singularity at $z = a$. Then in each neighborhood of a, f assumes each complex number, with one possible exception, an infinite number of times.

Proof. Without loss of generality, we take $a = 0$. ASSUME there is $R > 0$ such that there are two numbers not in $\{f(z) \mid 0 < |z| < R\}$. Without loss of generality we assume the two oriented values are 0 and 1 for $0 < |z| < R$ (otherwise, we can replace $f(z)$ with $(f(z) - a)/(b - a)$ where a and b are the omitted values).

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Theorem XII.4.2 (continued 1)

Proof (continued). Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that $f_{n_k} \to \varphi$ in $C(G, \mathbb{C}_\infty)$. Then by Proposition VII.1.10(b), $f_{n_k} \to \varphi$ uniformly on compact subsets of G and so $f_{n_k} \to \varphi$ uniformly on $\{z \mid |z| = R/2\}$. Since each f_{n_k} is analytic on G and the convergence is in $C(G, \mathbb{C}_\infty)$, then either φ is analytic or $\varphi \equiv \infty$ (the uniform convergence on $|z| = R/2$ implies continuity of φ so if $\varphi - \infty$ at any point then it must be ∞ throughout $G = B(0; R) \setminus \{0\}$). ASSUME φ is analytic. Let $M = \max\{|\varphi(z)| \mid |z| = R/2\}$. Then $|f(z/n_k)| = |f_{n_k}(z)| = |f_{n_k}(z) = \varphi(z) + \varphi(z)| \leq |f_{n_k}(z) - \varphi(z)| + |\varphi(z)| \leq 2M$ for n_k sufficiently large and $|z| = R/2$. Thus $|f(z)| \leq 2M$ for $|z| = R/(2n_k)$ and for sufficiently large n_k (say $k \geq K$). So $|f(z)| \leq 2M$ on $\text{ann}(0; R/(2n_{k_1}), R/(2n_{k_2}))^-$ for all $k_1 > k_2 > K$ by the Maximum Modulus Theorem–2nd Version (Theorem VI.1.2).

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Suppose an analytic function f has an essential singularity at $z = a$. Then in each neighborhood of a, f assumes each complex number, with one possible exception, an infinite number of times.

Proof (continued). But then, by Exercise XII.4.B, f must have a pole at $z = 0$, again a CONTRADICTION. So the original assumption that f omits two values is false and f cannot omit two values.

To show that f assumes every value an infinite number of times (with one possible exception). Suppose some complex w is assumed in $G \subset B(0; R) \setminus \{0\}$ only finitely many times (so w is not the possibly exceptional point). Then we can find a z value of smallest modulus at which f takes on the value w, and then repeat the above argument to show that there is a punctured disk in which f fails to assume two values which yields a contradiction. □

Theorem XII.4.2 (continued 2)

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Corollary XII.4.4

Corollary XII.4.4. If f is an entire function that is not a polynomial then f assumes every complex number, with one possible exception, an infinite number of times.

Proof. Consider $g(z) = f(1/z)$. Since f is not a polynomial then the power series for f contains infinitely many nonzero coefficients and so g has an essential singularity at $z = 0$.

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Proof. Consider $g(z) = f(1/z)$. Since f is not a polynomial then the power series for f contains infinitely many nonzero coefficients and so g has an essential singularity at $z = 0$. By the Great Picard Theorem (Theorem XII.4.2), g assumes every complex number with one possible exception. So f has the same property (if $w = g(z)$ where $z \neq 0$ then $f(1/z) = w$). □

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