A Primer on Lipschitz Functions

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Note. The purpose of these notes is to contrast the behavior of functions of a real variable and functions of a complex variable. Recall that a function of a complex variable which is continuously differentiable on a ball of center a and radius $R > 0$ has a power series representation on $B(a; R)$ (Theorem IV.2.8 of Conway). Of course, a function of a real variable can be differentiable but not twice differentiable. Or it can be twice differentiable, but not three times differentiable. Even worse, a function of a real variable can be infinitely differentiable but not have a power series representation, as shown by the classic example:

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

This function is infinitely differentiable at $x = 0$ and $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$. But then f cannot have a power series representation which is valid on an open interval containing 0. These comments about functions of a real variable imply that the set of n-times differentiable functions are a proper subset of the set of $(n+1)$-times differentiable functions and that the set of infinitely differentiable functions is a proper subset of the set of functions with a power series representation. In fact, these sets can be further divided using the property of “Lipschitz.” This allows us to create an infinite chain of sets of functions of a real variable, starting with continuous functions and ending with functions with a power series representation. In the setting of functions of a complex variable, there is no such chain since a function which is continuously differentiable has a power series representation and the chain of sets of functions in the complex setting is of length only two or three.

Note. Since we are interested in contrasting functions of a real variable and functions of a complex variable, we sometimes consider real functions (when speaking of differentiation), but when possible we consider functions $f : X \to \Omega$ where (X, d) and (Ω, ρ) are metric spaces (when speaking of the property of Lipschitz). When we use the expression "function f is analytic" in these notes, we mean that f has a power series representation over the set under consideration.

Definition. Function f is *Lipschitz* on X if there exists $M \in \mathbb{R}$ such that $\rho(f(x), f(y)) \leq M\, d(x, y)$ for all $x, y \in X$; M is a *Lipschitz constant* for f on X. Function f is *locally Lipschitz* on $W \subset X$ if for each $w \in W$ there exists open $W_0 \subset W$ containing w such that f is Lipschitz on W_0.

Theorem 1. If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then f is locally Lipschitz.

Proof. Let $x, y \in \mathbb{R}$ where, say, $x < y$. The by the Fundamental Theorem of Calculus (this is where we need f to have a *continuous* derivative) we have

$$f(x) - f(y) = f(y + (1)(x - y)) - f(y + (0)(x - y))$$

$$= \int_0^1 \frac{d}{dt}[f(y + t(x - y))]\, dt = \int_0^1 f'(y + t(x - y))(x - y)\, dt.$$

So

$$|f(x) - f(y)| = \left| \int_0^1 f'(y + t(x - y))(x - y)\, dt \right|$$

$$\leq \int_0^1 |f'(y+t(x-y))|\, |x-y|\, dt = \int_0^1 |f'(y+t(x-y))|\, dt\, |x-y| \leq M|x-y|$$

where $\max_{0 \leq t \leq 1} |f'(y + t(x - y))| = \max_{x \leq u \leq y} |f'(u)| = M$. So for any $w \in \mathbb{R}$ we can take $W_0 = (w - 1, w + 1)$ and f is Lipschitz on W_0 with Lipschitz constant $\max_{w-1 \leq u \leq w+1} |f'(u)|$. Hence, f is locally Lipschitz, as claimed. ∎

Theorem 2. If $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is differentiable on I, then f is Lipschitz on I if and only if f' is bounded on I (this is Theorem 9.5.1 of Searcóid).

Proof. First, suppose f' is bounded on I, say by M: $|f'(x)| \leq M$ for all $x \in I$. Let x and y be any two distinct elements of I. Then by the Mean Value Theorem, there exists w between x and y such that $f'(w) = \dfrac{f(x) - f(y)}{x - y}$. Since $|f'(w)| \leq M$, then $|f(x) - f(y)| \leq M|x - y|$. Therefore f is Lipschitz on I with Lipschitz constant M.

Second, suppose f' is not bounded on I. Let $r \in \mathbb{R}$ be arbitrary. Then for some distinct $x, y \in I$ we have that the difference quotient satisfies $\left|\dfrac{f(x) - f(y)}{x - y}\right| > r$, or that $|f(x) - f(y)| > r|x - y|$. Since r is arbitrary, then f is not Lipschitz. ∎

Theorem 3. If f is locally Lipschitz on X and X is compact, then f is Lipschitz on X.

Proof. Since f is locally Lipschitz on X, for each $x \in X$ there exists an open W_x containing x such that f is Lipschitz on W_x. Consider the collection of all such W_x. This collection forms an open cover of X and so there is a finite subcollection $\{W_1, W_2, \ldots, W_n\}$ which also covers X, since X is compact. Since f is Lipschitz on each W_i, there is an M_i such that $\rho(f(x_i), f(y_i)) \leq M_i\, d(x_i, y_i)$ for all $x_i, y_i \in W_i$, for $i \in \{1, 2, \ldots, n\}$. Taking $M = \max\{M_1, M_2, \ldots, M_n\}$, we see that f is Lipschitz on X. ∎

Theorem 4. If $f : X \to \Omega$ is Lipschitz on X then f is uniformly continuous on X (this is Theorem 9.4.2(i) of Searcóid.) If f is locally Lipschitz on X then f is continuous on X.

Proof. Both results follow from the ε and δ definitions where we take $\delta = \varepsilon/M$ where M is the Lipschitz constant on X or at point $x \in X$. $\square$

Theorem 5. If $f : X \to \Omega$ is continuous on X and X is compact, then f is uniformly continuous on X. (Theorem II.5.15 of Conway.)

Proof. This is a standard result from the senior level Analysis 1 (MATH 4217/5217) class. $\square$

Example 1. If f is continuous on X, then f need not be uniformly continuous on X. Consider $f(x) = 1/x$ on $X = (0, 1)$.

Example 2. If f is uniformly continuous on X, then f need not be Lipschitz on X. Consider $f(x) = \sqrt{1 - x^2}$ on $X = [-1, 1]$. Since f' is not bounded, then by Theorem 2 f is not Lipschitz on X.

Example 3. If f is locally Lipschitz on X, then f need not be Lipschitz on X. Consider $f(x) = 1/x$ on $X = (0, 1)$. f is continuously differentiable on $X = (0, 1)$ and so is locally Lipschitz on X by Theorem 1. However,

$$\frac{f(x) - f(y)}{x - y} = \frac{-1}{xy}$$

can be made arbitrarily large by making x and y near 0, and so f is not Lipschitz on $X = (0, 1)$.

Example 4. If f is locally Lipschitz on X, then f need not be differentiable on X. Consider

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \in [-1,0) \cup (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

and $X = [-1,1]$. Then $f'(x) = \sin(1/x) - 1/x\cos(1/x)$ for $x \in [-1,0) \cup (0,1]$ and $f'(0)$ does not exist (consider $x \to 0$). So f is locally Lipschitz on $[-1,0) \cup (0,1]$ by Theorem 1. Consider

$$-1 \leq \frac{f(0) - f(y)}{0 - y} = \sin(1/y) \leq 1.$$

Then $|f(0) - f(y)| \leq 1 \times |0 - y|$ and so f is Lipschitz at $x = 0$ with Lipschitz constant $M = 1$, and therefore f is locally Lipschitz on $X = [-1,1]$.

Note. If $f : \mathbb{R} \to \mathbb{R}$ then we have the following (pointwise) implications:

$$f \text{ continuously differentiable} \Rightarrow f \text{ locally Lipschitz} \Rightarrow f \text{ continuous.}$$

If $f : \mathbb{R} \to \mathbb{R}$ and X is compact, then we have the following implications on set X:

$$f \text{ continuously differentiable} \Rightarrow f \text{ locally Lipschitz} \iff f \text{ Lipschitz}$$

$$\Rightarrow f \text{ uniformly continuous} \iff f \text{ continuous.}$$

Each of the one way implications given here cannot be reversed, as is shown by examples above.

Note. It is common to study sets of functions defined on compact sets. For example, the set of all continuous (real) functions defined on $[0,1]$ forms the set denoted $C^0([0,1])$. This set is closed under linear combinations (i.e., if $f, g \in C^0([0,1])$ then $\alpha f + \beta g \in C^0([0,1])$ where α, β are real constants). Therefore this set forms a "linear space." In fact, a norm can be put on this space and Cauchy sequences of functions on $C^0([0,1])$ ("C^0 functions" for short) converge to C^0 functions (i.e., the space is complete—the norm of $f \in C^0$ is $\|f\| = \max\{|f(x)| \mid x \in [0,1]\}$). If f is n times differentiable and $f^{(n)}$ is continuous on $[0,1]$, then f is in $C^n([0,1])$ and f is said to be "C^n." Infinitely differentiable functions are said to be C^∞. Recall that a function is *analytic* on a set if it has a power series representation at each point of that set, and analytic functions must be infinitely differentiable. We therefore have the following inclusions:

$$\{f \mid f \text{ is analytic}\} \subset C^\infty \subset \cdots \subset C^n \subset$$

$$\cdots \subset C^2 \subset C^1 \subset C^0 = \{f \mid f \text{ is continuous}\}.$$

As we have seen above, there is another class of related functions called *Lipschitz functions*. If f if Lipschitz on $[0,1]$, then we say f is $\text{Lip}([0,1])$ and if $f^{(n)}$ is Lipschitz on $[0,1]$ then we say that $f \in \text{Lip}^{(n+1)}([0,1])$. (Since we are considering functions on compact sets, there is no need to consider "locally Lipschitz" or "uniformly continuous.") Therefore $C^{n+1} \subset \text{Lip}^{n+1} \subset C^n$ for all $n \in \mathbb{N}$. So our chain of sets becomes refined to:

$$\{f \mid f \text{ is analytic}\} \subset C^\infty \subset \cdots \subset C^{n+1} \subset \text{Lip}^{n+1} \subset C^n \subset$$

$$\cdots \subset \text{Lip}^3 \subset C^2 \subset \text{Lip}^2 \subset C^1 \subset \text{Lip} \subset C^0 = \{f \mid f \text{ is continuous}\}.$$

Note. For an open set $U \subset \mathbb{C}$, here is the corresponding sets of complex valued functions on U:

$$\{f \mid f \text{ is analytic}\} = C^\infty = \cdots = C^2 = C^1 \subset C^0 = \{f \mid f \text{ is continuous}\},$$

or more briefly

$$\{f \mid f \text{ is analytic}\} = C^1 \subset C^0 = \{f \mid f \text{ is continuous}\},$$

or more briefly still:

$$C^1 \subset C^0.$$

Note. So there are dramatic differences in the behaviors of functions of a real variable and functions of a complex variable! One way you might informally read this is that it is "very hard" for a function of a complex variable to be continuously differentiable. So much so, that if it satisfies this condition then it has a power series representation!

References

1. Conway, John B., *Functions of One Complex Variable I, Second Edition*, *Graduate Tests in Mathematics*, New York: Spring Science+Business Media, 1978.

2. Searcóid, Micheál, *Metric Spaces*, *Springer Undergraduate Mathematics Series*, London: Springer-Verlag London Limited, 2007.

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