The Ilieff-Sendov Conjecture

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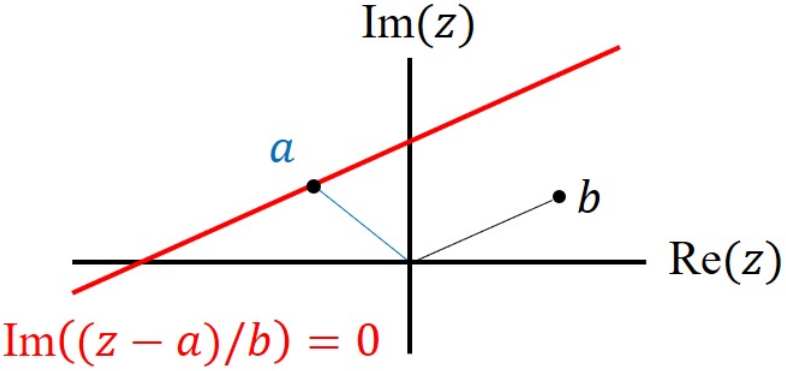
Note. Recall that a line in the complex plane can be represented by an equation of the form $\text{Im}((z-a)/b) = 0$ where the line is "parallel" to the vector b and translated from the origin by an amount a (here we are knowingly blurring the distinction between vectors in $\mathbb{R}^2$ and numbers in $\mathbb{C}$).



We can represent a closed half plane with the equation $\text{Im}((z-a)/b) \leq 0$. This represents the half plane to the right of the line $\text{Im}((z-a)/b) = 0$ when traveling along the line in the "direction" of b. This representation, along with some standard properties of logarithms and derivatives in the complex setting, allow us to prove the following so-called Gauss-Lucas Theorem (or sometimes simply the Lucas Theorem). For a reference, see page 29 of [1].

Theorem 1. The Gauss-Lucas Theorem.
If all the zeros of a polynomial $P(z)$ lie in a half plane in the complex plane, then all the zeros of the derivative $P'(z)$ lie in the same half plane.

Proof. By the Fundamental Theorem of Algebra, we can factor P as

$$P(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n).$$

So

$$\log P(z) = \log a_n + \log(z - r_1) + \log(z - r_2) + \cdots + \log(z - r_n)$$

and differentiating both sides gives

$$\frac{P'(z)}{P(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2} + \cdots + \frac{1}{z - r_n} = \sum_{k=1}^{n} \frac{1}{z - r_k}. \quad (*)$$

Suppose the half plane H that contains all the zeros of $P(z)$ is described by $\mathrm{Im}((z - a)/b) \leq 0$. Then $\mathrm{Im}((r_k - a)/b) \leq 0$ for $k = 1, 2, \ldots, n$. Now let z^* be some number not in H. We want to show that $P'(z^*) \neq 0$ (this will mean that all the zeros of $P'(z)$ are in H). Well, $\mathrm{Im}((z^* - a)/b) > 0$. Let r_k be some zero of P. Then

$$\begin{aligned}\mathrm{Im}\left(\frac{z^* - r_k}{b}\right) &= \mathrm{Im}\left(\frac{z^* - a - r_k + a}{b}\right) \\ &= \mathrm{Im}\left(\frac{z^* - a}{b}\right) - \mathrm{Im}\left(\frac{r_k - a}{b}\right) > 0.\end{aligned}$$

(Notice that $\mathrm{Im}((z^* - a)/b) > 0$ since z^* is not in H, and $-\mathrm{Im}((r_k - a)/b) \geq 0$ since r_k is in H.) The imaginary parts of reciprocal numbers have opposite signs, so $\mathrm{Im}(b/(z^* - r_k)) < 0$. Hence, by applying $(*)$,

$$\mathrm{Im}\left(\frac{bP'(z^*)}{P(z^*)}\right) = \sum_{k=1}^{n} \mathrm{Im}\left(\frac{b}{z^* - r_k}\right) < 0.$$

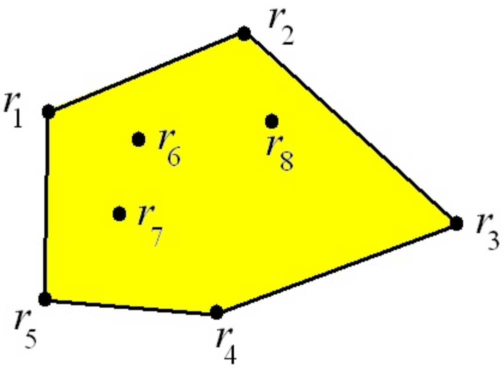
So $\dfrac{P'(z^*)}{P(z^*)} \neq 0$ and $P'(z^*) \neq 0$. Therefore, if $P'(z) = 0$, then $z \in H$.

□

Note. With repeated application of the Gauss-Lucas Theorem, we can prove the following corollary.

Corollary 1. *The convex polygon in the complex plane which contains all the zeros of a polynomial P, also contains all the zeros of P'.*

Note. For example, if P has eight zeros, then the convex polygon containing them might look like the following.



Note. There is no "clean" analogy of the Gauss-Lucas Theorem in the real setting (which is the case with many complex analysis results). For example, can you find a real polynomial with all of its zeros in a certain interval $[a, b]$, yet it has critical points outside of the interval? The answer: YES!

Note. A weaker version of Corollary 1 is the following.

Corollary 2. *A circle which contains all of the zeros of polynomial P, also contains all of the zeros of P'.*

Note. We now restrict our study to a class of polynomials which is sort of "normalized" with respect to the location of zeros. This is the set of polynomials each of which has all of its zeros in the unit disk of the complex plane, $\{z \mid |z| \leq 1\}$. We can easily use Corollary 2 to get a result for this class of polynomials.

Corollary 3. *If all the zeros of a polynomial* P *lie in* $|z| \leq 1$, *then all the zeros of* P' *also lie in* $|z| \leq 1$.

Note. It is important that we are studying the set of all *polynomials* with their zeros in $|z| \leq 1$. We can violate Corollary 3 by considering a non-polynomial. Consider, for example, $f(z) = ze^{z/2}$. Then the only zero of f is $z = 0$, and so all the zeros of f lie in $|z| \leq 1$. However, $f'(z) = e^{z/2} + \frac{1}{2}ze^{z/2} = (\frac{1}{2}z + 1)e^{z/2}$. Then f' has a zero at $z = -2$ and so there is a zero outside of $|z| \leq 1$.

Note. Another result concerning a relationship between the zeros of a polynomial and the zeros of the derivative involves the centroid of the zeros.

Definition. For polynomial $P(z) = a_n(z - r_1)(z - r_2) \cdots (z - r_n)$ with zeros r_1, r_2, $\ldots$, r_n (not necessarily distinct), define the *centroid of the zeros* by placing a unit mass at each r_k in the complex plane and then finding the center of mass of the resulting distribution (as in Calculus 2).

Theorem 2. *The centroid of the zeros of a polynomial* P *is the same as the centroid of the zeros of* P'.

Proof. Theorem 2 is listed as an exercise in Morris Marden's *Geometry of*

Polynomials (see Exercise 4, page 53, [9]). Let the zeros of n-degree polynomial P be $r_1, r_2, \ldots, r_n$, and so

$$P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - r_k).$$

Then $P(z) = a_n z^n + a_n(-r_1 - r_2 - \cdots - r_n)z^{n-1} + \cdots + a_1 z + a_0$. So $a_{n-1} = -a_n(r_1 + r_2 + \cdots + r_n)$ and the centroid of the zeros of P is

$$\frac{r_1 + r_2 + \cdots + r_n}{n} = \left(\frac{1}{n}\right)\left(\frac{-a_{n-1}}{a_n}\right) = \frac{-a_{n-1}}{na_n}.$$

Let the zeros of P' be $s_1, s_2, \ldots, s_{n-1}$. Then

$$P'(z) = \sum_{k=1}^{n} k a_k z^{k-1} = n a_n \prod_{k=1}^{n-1} (z - s_k).$$

Similar to above, the centroid of the zeros of P' is

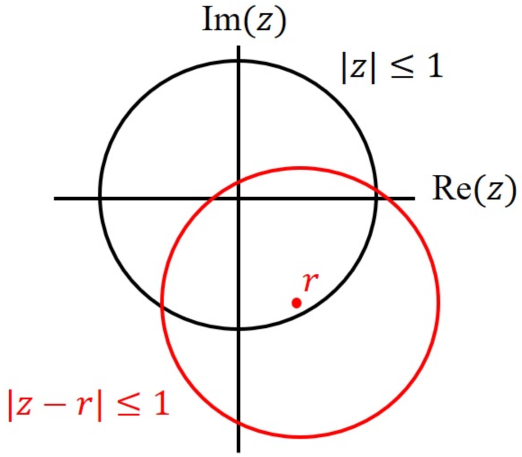
$$\frac{s_1 + s_2 + \cdots + s_{n-1}}{n-1} = \left(\frac{1}{n-1}\right)\left(\frac{-(n-1)a_{n-1}}{na_n}\right) = \frac{-a_{n-1}}{na_n}.$$

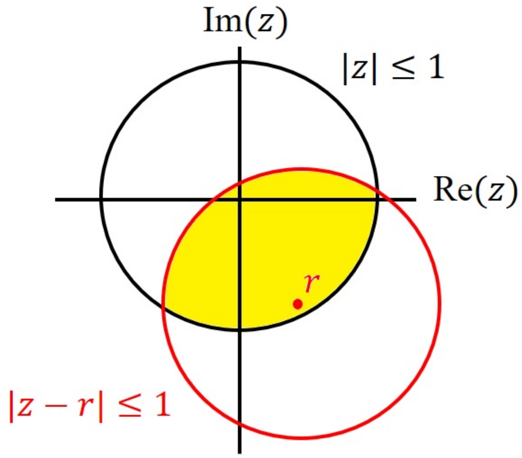
Therefore, the centroid of the zeros of P and the centroid of the zeros of P' are the same. □

Note. Now for the object of our interest! It is known variously as the Ilieff Conjecture, the Ilieff-Sendov Conjecture, and the Sendov Conjecture (making it particularly difficult to search for papers on the subject). It was originally posed by Bulgarian mathematician Blagovest Sendov in 1958 (according to [12]; sometimes the year 1962 is reported [11]), but often attributed (as Miller says in [11]) to Ilieff because of a reference in Hayman's *Research Problems in Function Theory* in 1967 [8].

Conjecture. The Ilieff-Sendov Conjecture.
If all the zeros of a polynomial P lie in $|z| \leq 1$ and if r is a zero of P, then there is a zero of P' in the circle $|z - r| \leq 1$.

Note. Combining the Ilieff-Sendov Conjecture with Corollary 3, we can further restrict the conjectured location of the critical points of P.





A Personal Note. I first saw a presentation on this conjecture in April 1986 at the "12th Annual Graduate Mathematics Conference" at Syracuse University. The presentation was by Michael Miller of LeMoyne College in Syracuse, NY. It was (in some part) this presentation that drew me away from discrete math and towards complex analysis for my Ph.D. topic.

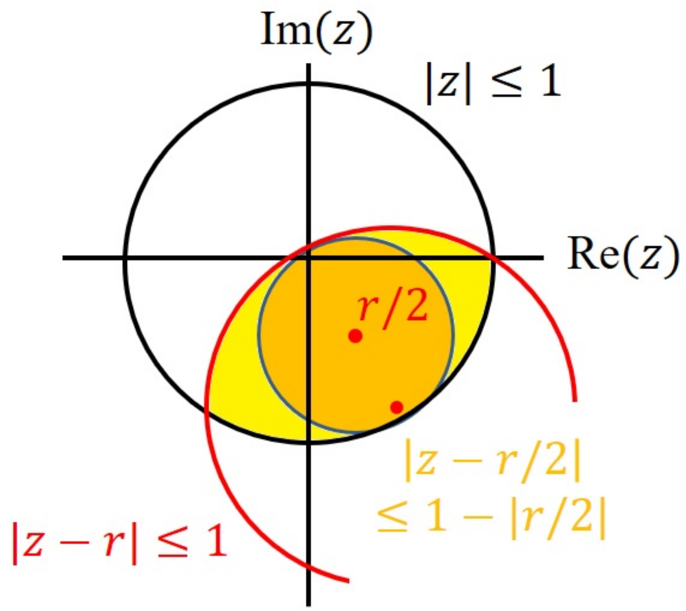
Note. According to a 2008 paper by Michael Miller [12], there have been over 80 papers written on the conjecture. As a result, it has been demonstrated in many special cases. Some of the special cases are (where we understand that all polynomials have their zeros in $|z| \leq 1$):

1. 3rd and 4th degree polynomials [15],
2. 5th degree polynomials [10],
3. all zeros of a polynomial p of modulus 1 have a zero of p' within distance 1 [15],

4. If the convex hull containing the roots of a polynomial p have its vertices on $|z| = 1$ then p satisfies the conjecture [17],
5. polynomials with real and non-positive coefficients [18],
6. polynomials with at most three distinct zeros [6, 16],
7. polynomials with at most six distinct zeros [3],
8. polynomials of degree less than or equal to 6 [4],
9. polynomials of degree less than or equal to 8 [5], and
10. the circle $|z - r| \leq 1.08331641$ [2].

Note. Many of the papers on the conjecture have been a bit more computational and have centered around finding "extremal" polynomials which push the locations of the zero of P to the edge of the $|z - r| \leq 1$ region. This is especially true of the work of Micheal Miller [11, 12].

Note. A common approach to proving a difficult conjecture is to prove something even more restrictive than the conjecture, and then the conjecture follows as a corollary. This is how Andrew Wiles eventually gave a proof of Fermat's Last Theorem—he proved the more general " Taniyama-Shimura Conjecture" for semistable elliptic curves, and from this Fermat's Last Theorem followed. In 1969, Goodman, Rahman, and Ratti [7] (and independently Schmiesser [17]) conjectured that the Ileff-Sendov Conjecture could be modified to the claim that (with the notation above) the region $|z-r/2| \leq 1-|r|/2$ must contain a zero of P'. This is the orange region here:

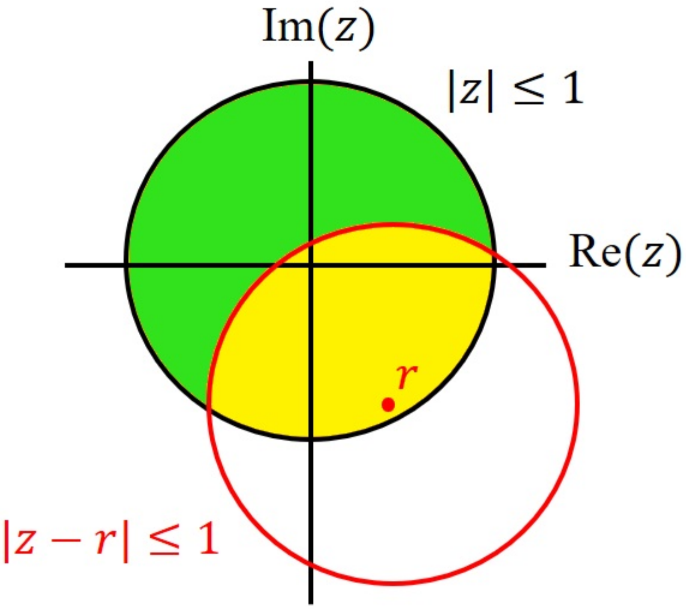


However, this conjecture is not true as shown by Micheal Miller in 1990 [11]. The following eighth degree polynomial violates the Goodman, Rahman, Ratti Conjecture:

$$P(z) = (z - 0.8)(z^7 + 1.241776468z^6 + 1.504033112z^5 + 1.702664563z^4$$
$$+1.702664563z^3 + 1.504033112z^2 + 1.241776468z + 1).$$

Miller also found degree 6, 10, and 12 polynomials violating the new conjecture.

Dr. Bob's Speculation. In order to find a counterexample to the Ilieff-Sendov Conjecture, we would need to find a polynomial for which there is a zero of P at some point r, yet the region (in yellow in the figure below) is free of zeros of P'. This would force the zeros of P' to lie in the "lune-shaped" region (green below). However, if this is the case then the centroid of the zeros of P must lie rather far from r. *It seems to me* that this could not happen in light of Theorem 2 and the location of r. To my knowledge, no one has taken this approach…maybe with good reason!



Note. Though this conjecture has interested me for decades, I did not publish anything on it until 2020. The paper is coauthored with Ghulam Sofi and Shabir Ahanger of Central University of Kashmir, India [20]. The results in this paper are based on a "useful relationship" between the zeros and critical points of a polynomial. Suppose p is a polynomial of degree n with zeros $r_1, r_2, \ldots, r_n$ and leading coefficient a_n so that $p(z) = a_n \prod_{k=1}^{n}(z - r_k)$. Let the zeros of p' be ζ_k for $k = 1, 2, \ldots, n-1$, and then

$$p'(z) = na_n \prod_{k=1}^{n-1}(z - \zeta_k). \tag{1}$$

We have by the product rule that

$$p'(z) = a_n \frac{d}{dz}\left[\prod_{k=1}^{n}(z - r_k)\right] = a_n \sum_{j=1}^{n} \prod_{k=1,k\neq j}^{n} (z - r_k).$$

So for any j with $1 \leq j \leq n$ we have $p'(r_j) = a_n \prod_{k=1,k\neq j}^{n} (r_j - r_k)$. Also

$p'(r_j) = na_n \prod_{k=1}^{n-1}(r_j - \zeta_k)$ by (1) and therefore

$$na_n \prod_{k=1}^{n-1}(r_j - \zeta_k) = a_n \prod_{k=1, k\neq j}^{n} (r_j - r_k),$$

so that we have the "useful relationship"

$$n \prod_{k=1}^{n-1} |r_j - \zeta_k| = \prod_{k=1,k\neq j}^{n} |r_j - r_k| \text{ for } 1 \leq j \leq n. \tag{2}$$

Notice that the left-hand side of (2) involves distances between zeros and critical points, and the right-hand side involves distances between zeros. If $\prod_{k=1}^{n-1} |r_j - \zeta_k| \leq 1$ for $1 \leq j \leq n$ then Sendov's conjecture holds for polynomial p. The useful relationship is used in [20] to prove that the Sendov conjecture holds for polynomials falling into certain classes. One of the results is the following.

Theorem A. Let $p(z) = a_n \prod_{k=1}^{n}(z - r_k)$ be a polynomial of degree $n \geq 2$ with its zeros satisfying $|r_k| \leq 1$ for $k = 1, 2, \ldots, n$ and with

$$|a_n| \geq \frac{n+1}{2n} \max_{|z|=1} |p(z)| = \frac{n+1}{2n} \|p\|.$$

Then each of the disks $|z - r_k| \leq 1$, for $k = 1, 2, \ldots, n$, must contain a zero of $p'(z)$. That is, p satisfies Sendov's conjecture.

A copy of [20] is on my webpage: Some Classes of Polynomials Satisfying Sendov's Conjecture.

Note. Another recent result (2021) by Sofi and Shah concerns a special class of polynomials [19]:

Theorem B. Let $p(z) = a_n \prod_{k=1}^{n}(z - r_k)$ be a polynomial of degree $n \geq 2$ with its zeros satisfying $|r_k| \leq 1$ for $k = 1, 2, \ldots, n$. Suppose all z_k for $1 \leq k \leq n$ lie on a circle or on a line within the closed disk. Then the Sendov conjecture holds for p.

Note. So far, we have mentioned results that show that the Sendov conjecture holds either for certain small degree polynomials or that it holds for a class of polynomials satisfying some special condition. Well-known mathematician Terence Tao of the University of California, Los Angles has presented a manuscript, "Sendov's Conjecture for Sufficiently High Degree Polynomials," claiming the following result:

Theorem C. Sendov's conjecture is true for all sufficiently large n. That is, there exists an absolute constant n_0 such that Sendov's conjecture holds for $n \geq n_0$.

Notice that the value of n_0 is not given, but only its existence is claimed.

Breaking News! In April 2021, Petar P. Petrov presented a manuscript claiming to give a proof of Sendov's conjecture, "Analyticity Domains of Critical Points of Polynomials, A Proof of Sendov's Conjecture" [13]. Rahman and Schmeisser commented in their 2002 *Analytic Theory of Polynomials* [14]:

> "After more than thirty years of research on Sendov's conjecture, it seems that the standard methods from the theory of polynomials have been exhausted and new approaches are needed."

Petrov has followed what might be called a "nonstandard method" in that he has applied the theory of functions of several complex variables and used the Implicit Mapping Function. A copy of this manuscript is online at: https://arxiv.org/abs/2104.00348. As of the writing of these notes (January 2022), this manuscript has not yet been through the complete peer review process and is not yet published.

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