

# Ordering the Complex Numbers

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**Note.** Recall (from Kirkwood) that an ordered field  $\mathbb{F}$  is:

**Definition.** A field  $\mathbb{F}$  is *ordered* if there is a nonempty set  $P \subset \mathbb{F}$  (called the *positive subset*) for which

(1)  $a, b \in P$  implies  $a + b \in P$

(2)  $a, b \in P$  implies  $ab \in P$

(3)  $a \in \mathbb{F}$  implies exactly one of the following:  $a \in P$ ,  $-a \in P$ , or  $a = 0$ .

For  $a, b \in \mathbb{F}$ , if  $b - a \in P$ , then we say  $a < b$ .

**Note.** Property (1) is closure under addition, Property (2) is closure under multiplication, and Property (3) is the Law of Trichotomy.

**Theorem 1.** If  $a < b$  and  $c > 0$ , then  $ac < bc$ .

**Proof.** Since  $a < b$ , then  $b - a \in P$ . Since  $c > 0$ , then  $c \in P$ . By Property (2),  $(b - a)c = bc - ac \in P$  and so  $ac < bc$ . ■

**Theorem 2.**  $0 < 1$

**Proof.** Suppose not. Then  $1 < 0$  (every field has at least two distinct elements: the additive identity, 0, and the multiplicative identity, 1 — so  $0 \neq 1$ ). Then  $1 \notin P$  and so  $-1 \in P$ . But then  $(-1)(-1) \in P$  and  $(-1)(-1) = 1$  (a property of fields), a contradiction. Therefore,  $0 < 1$ . ■

**Note.** Suppose  $\mathbb{C}$  is an ordered field. We take  $i$  to satisfy  $i^2 = (i)(i) = -1$ . By the Law of Trichotomy, either  $i \in P$ ,  $-i \in P$ , or  $i = 0$  (but  $0^2 = 0 \neq 1$ , so  $i \neq 0$ ).

**Corollary 1.**  $i \notin P$ .

**Proof.** Suppose to the contrary that  $i \in P$ . Then  $(i)(i) = i^2 = -1 \in P$ . But this contradicts Theorem 2. So  $i \notin P$ . ■

**Corollary 2.**  $-i \notin P$ .

**Proof.** Suppose to the contrary that  $-i \in P$ . Then  $(-i)(-i) = (i)(i) = i^2 = -1 \in P$ , Again, a contradiction to Theorem 2. So  $-i \notin P$ . ■

**Theorem 3.**  $\mathbb{C}$  is not an ordered field.

**Proof.** By Corollaries 1 and 2,  $i$  does not satisfy the Law of Trichotomy, so  $\mathbb{C}$  is not an ordered field. ■

**Note.** The above result does not imply that we cannot put *any* sort of ordering on  $\mathbb{C}$ ! Only that (in informal terms) we cannot put a *useful* ordering on  $\mathbb{C}$ . Consider  $z_1 = a_1 + ib_1$  and  $z_2 = a_2 + ib_2$ . Define  $z_1 \prec z_2$  if and only if either (1)  $a_1 < a_2$  or (2)  $a_1 = a_2$  and  $b_1 < b_2$ . This is called the *lexicographic* ordering of  $\mathbb{C}$ . This is because it is similar to the way words are alphabetized. For any  $z_1, z_2 \in \mathbb{C}$  we have exactly one of the following: (1)  $z_1 \prec z_2$ , (2)  $z_2 \prec z_1$ , or (3)  $z_1 = z_2$ . Also, if  $z_1 \prec z_2$  and  $z_2 \prec z_3$ , then  $z_1 \prec z_3$ .

**Note.** With the lexicographic ordering,  $0 \prec i$  (i.e.,  $i$  is “positive; if such a thing makes sense). With  $a = 0$ ,  $b = i$ , and  $c = i$ , we have:  $a \prec b$  and  $0 \prec c$ , but  $ac = 0 \prec bc = (i)(i) = i^2 = -1$  which is not the case! So Theorem 2 does not hold for the lexicographic ordering of  $\mathbb{C}$ .

**Note.** The lexicographic ordering is also useless in defining completeness. Consider the set  $A = \{z \in \mathbb{C} \mid z \prec 0\}$ . The set  $A$  has an  $\prec$ -upper bound (say 1), and has a  $\prec$ -least upper bound (namely 0). Consider the set  $B = \{z = a + ib \mid a \leq 0 \text{ and } b \in \mathbb{R}\}$ . Then  $B$  has an  $\prec$ -upper bound (say 1), but  $B$  has no  $\prec$ -least upper bound!

**Note.** Perhaps you are familiar with the idea of a *well ordering* from set theory. This is in no way related to the type of ordering we are talking about! An *ordering* of a set  $A$  is a binary relation  $\preceq$  on the set such that the relation is (1) reflexive ( $a \preceq a$  for all  $a \in A$ ), (2) antisymmetric ( $a \preceq b$  and  $b \preceq a$  imply  $a = b$ ), and (3) transitive ( $a \preceq b$  and  $b \preceq c$  imply  $a \preceq c$ ). An ordering  $\preceq$  of a set  $A$  is a *total ordering* if for any  $a, b \in A$ , either  $a \preceq b$  or  $b \preceq a$  (that is, any two elements of  $A$  are *comparable*). An example of a totally ordered set is the set  $A = \mathbb{R}$  with ordering  $\leq$ .

**Note.** A total ordering  $\preceq$  of a set  $A$  is a *well-ordering* if every nonempty subset of  $A$  has a  $\preceq$ -least element. That is, if  $B \subset A$  then there exists  $m \in B$  such that  $m \preceq b$  for all  $b \in B$ . An example of a well-ordered set is  $\mathbb{N}$  with ordering  $\leq$ . Notice that  $\leq$  does not well-order  $\mathbb{R}$ .

**Note.** The *Well-Ordering Principle* states that every set can be well-ordered. This result is equivalent to the Axiom of Choice. It is therefore true that  $\mathbb{C}$  can be “well-ordered,” but this should not be confused with the idea of ordering  $\mathbb{C}$  in a way that generalizes the ordering of  $\mathbb{R}$ .

**Note.** The lexicographic ordering of  $\mathbb{C}$  is an example of a total-ordering. However, as shown above, NO ordering of  $\mathbb{C}$  can satisfy the Law of Trichotomy, and hence there is no (useful) way to extend the ordering of  $\mathbb{R}$  to an ordering of  $\mathbb{C}$ .

## References

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