Supplement. Location of Zeros of Polynomials

Note. The Fundamental Theorem of Algebra states that a polynomial of degree n with complex coefficients has n zeros (counting multiplicity). However, this theorem does not say anything about *what* or *where* the zeros are. For polynomials of degree 4 or less, there are algebraic formulae for the zeros. There are also analytic techniques to precisely find zeros in certain cases; for example, the Wolfram software company (the developers of Mathematica) has produced a poster which explains how to analytically solve an arbitrary 5th degree polynomial equation; see the Solving the Quintic Poster Wolfram page (accessed 1/27/2022).

Note. The purpose of this supplement is to give several results concerning the location of the zeros of a polynomial with no information other than the coefficients of the polynomial. There is also a large body of work addressing this question, but with various restrictions on the coefficients (the classical Eneström-Kakeya Theorem and its generalizations, for example), but we do not cover these ideas here. The background knowledge for this supplement is exposure to the Triangle Inequality in $\mathbb{C}$ (Conway's Theorem I.3.A); any other background will be stated here.

Note. The results we present fall into two general categories: (1) those giving location of zeros explicitly in terms of the coefficients, and (2) those which give the location in terms of a zero of another polynomial.

Note. The classical results are due to Cauchy. They appear in his "Exercises de Mathématique," in *Oeuvres* (2) Vol. 9 (1829), page 122 [2]. Cauchy's result in the first category is as follows.

Theorem 1. Cauchy's Location of Zeros Theorem, Category (1).
If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| \leq 1 + \max_{0 \leq k < n} |a_k/a_n| = \max_{0 \leq k < n} \frac{|a_n| + |a_k|}{|a_n|}.$$

Note. The proof of Theorem 1 which we give is from Morris Marden's *Geometry of Polynomials* (see [8] page 123) .

Note. We can apply Theorem 1 to $z^n p(1/z)$ to get the following. The formal proof is left as Exercise I.3S.1.

Corollary 1. If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| \geq \min_{1 \leq k \leq n} \frac{|a_0|}{|a_0| + |a_k|}.$$

Note. We will need Descartes' Rule of Signs in the proof of Theorem 2, so we state it here:

> Let f be a polynomial with real coefficients, $f(z) = \sum_{k=0}^{n} a_k z^k$. The number of positive (real) roots of f is either equal to the number of sign differences between consecutive nonzero coefficients in the sequence $a_0, a_1, \ldots, a_n$ or less than this number by an even number.

Note. The second classical result due to Cauchy also appeared in "Exercises de Mathématique," in *Oeuvres* (2) Vol. 9 (1829), page 122 [2]. Cauchy's result in the second category is as follows.

Theorem 2. Cauchy's Location of Zeros Theorem, Category (2).
If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in $|z| \le r$, where r is the positive root of the equation

$$|a_n|x^n - (|a_{n-1}|x^{n-1} + |a_{n-2}|x^{n-2} + \cdots + |a_1|x + |a_0|) = 0.$$

Note. In 1849, following his exploration of the Fundamental Theorem of Algebra, Carl Friedrich Gauss (1777–1855) published the next result [5]. Gauss is famous for not publishing his work, so this is likely one of many related results of his (see [1]).

Theorem 3. Gauss' Location of Zeros Theorem, Category (2).

If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in $|z| \leq r$, where r is the positive root of the equation

$$z^n - \sqrt{2}(|a_1|x^{n-1} + |a_2|x^{n-2} + \cdots + |a_{n-1}|x + |a_n|) = 0.$$

Note. Hölder's Inequality in $\mathbb{R}^n$ states that for $\vec{a} = [a_1, a_2, \ldots, a_n]$ and $\vec{b} = [b_1, b_2, \ldots, b_n]$ in $\mathbb{R}^n$ we have

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q}$$

where $1/p + 1/q = 1$, $p > 1$, and $q > 1$. If you are familiar with the normed linear spaces ℓ^p, this states that $\sum_{k=1}^{n} |a_k b_k| \leq \|\vec{a}\|_p \|\vec{b}\|_q$ (in fact, $\sum_{k=1}^{n} |a_k b_k|$ is the ℓ^1 norm of $[a_1b_1, a_2b_2, \ldots a_nb_n]$). The following result, due (independently) to Kuniyeda, Montel, and Tôya (see [8] for references), is based on this version of Hölder's Inequality.

Theorem 4. Kuniyeda, Montel, and Tôya.

For any p and q such that $1/p + 1/q = 1$, $p > 1$, and $q > 1$, all zeros of polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ lie in

$$|z| < \left\{ 1 + \left(\sum_{k=0}^{n-1} \frac{|a_k|^p}{|a_n|^p} \right)^{q/p} \right\}^{1/q} \leq \left(1 + n^{q/p} \left(\max_{0 \leq k \leq n-1} \frac{|a_k|}{|a_n|} \right)^q \right)^{1/q}.$$

Note. In 1967, Joyal, Labelle, and Rahman [7] proved a generalization of Cauchy's Theorem 1.

Theorem 5. Joyal, Labelle, Rahman Generalization of Theorem 1.
If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$|z| \le \frac{1}{2}\left(1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right)$$

where $B = \max_{0 \le k < n-1} |a_k/a_n|$.

Note. Theorem 5 is in fact an improvement of Theorem 1 since

$$\frac{1}{2}\left(1 + |a_{n-1}/a_n| + \sqrt{(1 - |a_{n-1}/a_n|)^2 + 4B}\right) \le 1 + \max_{0 \le k < n} |a_k/a_n|,$$

as is shown in Exercise I.3S.2.

Note. We can apply Theorem 5 to $z^n p(1/z)$ to get the following. The formal proof is left as Exercise I.3S.3.

Corollary 2. If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n with $a_0 \ne 0$, then no zeros of p lie in

$$|z| < 2 \Big/ \left(1 + |a_1/a_0| + \sqrt{(1 - |a_1/a_0|)^2 + 4\beta}\right)$$

where $\beta = \max_{1 \le k < n} |a_k/a_0|$.

Note. In 1978, Datt and Govil [3] give two related results, one in each of the two categories we have introduced.

Theorem 6. Datt and Govil, Category 2.

If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$\frac{|a_0|/|a_n|}{2(1+B)^{n-1}(Bn+1)} \leq |z| \leq 1 + \lambda B,$$

where $B = \max_{0 \leq k < n-1} |a_k/a_n|$ and λ is the unique root of the equation $x = 1 - 1/(1+Bx)^n$ in the interval $(0,1)$. The upper bound $1 + \lambda B$ is best possible and is attained for the polynomial $p(z) = z^n - B(z^{n-1} + z^{n-2} + \cdots + z + 1)$.

Note. Datt and Govil comment: "If we do not wish to look for the roots of the equation $x = 1 - 1/(1+Bx)^n$, we can still obtain a result which is an improvement of [Cauchy's Theorem 1]." With a proof similar to that they use for Theorem 6, they present the following.

Theorem 7. Datt and Govil, Category 1.

If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n, then all the zeros of p lie in

$$\frac{|a_0|/|a_n|}{2(1+B)^{n-1}(nB+1)} \leq |z| \leq 1 + \left(1 - \frac{1}{(1+B)^n}\right) B,$$

where $B = \max_{0 \leq k < n-1} |a_k/a_n|$ and λ is the unique root of the equation $x = 1 - 1/(1+Bx)^n$ in the interval $(0,1)$.

Note. Theorem 5 an improvement of Theorem 1, as shown in Exercise I.3S.2. However, a number of the published results in this area do not seem to address whether or not they are actually improvements over the theorems in the existing literature of the time. In 1990, V. K. Jain [6] presented two new results (one in each category), building on the work of Datt and Govil. He gave specific numerical examples showing that his results *could* give better bounds than the work of Datt and Govil. In 2012, Dehmer and Tsoy [4] published an extensive numerical study of the quality of the bounds on the zeros of polynomials for a large number of results in the literature.

References

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