Supplement. The Extended Complex Plane

Note. In Section I.6. 'The Extended Plane and Its Spherical Representation, we introduced the extended complex plane, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. We defined the function $d : \mathbb{C}_\infty \times \mathbb{C}_\infty \to \mathbb{R}$ as

$$\begin{aligned} d(z_1, z_2) &= \frac{2|z_1 - z_2|}{\{(1 + |z_1|^2)(1 + |z_2|^2\}^{1/2}} \text{ for } z_1, z_2 \in \mathbb{C} \\ d(z, \infty) &= \frac{2}{(1 + |z|^2)^{1/2}} \text{ for } z \in \mathbb{C}. \end{aligned}$$

Definition. A *metric space* is a pair (X, d) where X is a set and d is a function mapping $X \times X$ into $\mathbb{R}$ called a *metric* such that for all $x, y, z \in X$ we have

$$d(x, y) \geq 0$$

$$d(x, y) = d(y, x) \text{ (Symmetry)}$$

$$d(x, y) = 0 \text{ if and only if } x = y$$

$$d(x, z) \leq d(x, y) + d(y, z) \text{ (Triangle Inequality)}$$

For a given $x \in X$ and $r > 0$, define the *open ball* of center x and radius r as $B(x; r) = \{y \mid d(x, y) < r\}$. Define the *closed ball* of center x and radius r as $\overline{B}(x; r) = \{y \mid d(x, y) \leq r\}$.

Metric Theorem for $\mathbb{C}_\infty$. The function $d : \mathbb{C}_\infty \times \mathbb{C}_\infty \to \mathbb{R}$ defined above is a metric on $\mathbb{C}_\infty$.

Proof/Comment. It is easy to see that $d(z_1, z_2) \geq 0$, $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$, and $d(z_1, z_2) = d(z_2, z_1)$ for all $z_1, z_2 \in \mathbb{C}_\infty$. To prove that d satisfies the Triangle Inequality, we must show that

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C},$$

$$d(z_1, z_2) \leq d(z_1, \infty) + d(\infty, z_2) \text{ for all } z_1, z_2 \in \mathbb{C}, \text{ and}$$

$$d(z_1, \infty) \leq d(z_1, z_2) + d(z_2, \infty) \text{ for all } z_1, z_2, \in \mathbb{C}.$$

This is to be done in Exercise II.1.7. □

Note. So we have two metrics on $\mathbb{C}$; namely the metric induced by modulus, $d'(z_1, z_2) = |z_1 - z_2|$ (we denote this metric as $|\cdot|$), and the metric d defined on $\mathbb{C}_\infty$ (treating $\mathbb{C}$ and a metric subspace of $\mathbb{C}_\infty$).

Definition II.1.8/10. A set $G \subset X$ (where (X, d) is a metric space) is *open* if for all $x \in G$ there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset G$. A set $F \subset X$ is closed if $X \setminus F$ is open.

Definition II.4.1. A subset K of a metric space X is *compact* if for every collection $\mathcal{G}$ of open sets in X with the property $K \subset \cup_{G \in \mathcal{G}} G$, there is a finite number of sets $G_1, G_2, \ldots, G_n$ in $\mathcal{G}$ such that $K \subset \cup_{k=1}^{n} G_k$. The collection $\mathcal{G}$ is called an *open cover* of K.

Note. The next result is from Section VII.3, "Spaces of Meromorphic Functions,"

Proposition VII.3.3.

(a) If $a \in \mathbb{C}$ and $r > 0$, then there is $\rho > 0$ such that $B_\infty(a;\rho) \subset B(a;r)$.

(b) Conversely, if $\rho > 0$ is given and $a \in \mathbb{C}$ then there is a number $r > 0$ such that $B(a;r) \subset B_\infty(a;\rho)$.

(c) If $\rho > 0$ is given then there is a compact set $K \subset \mathbb{C}$ such that $\mathbb{C}_\infty \setminus K \subset B_\infty(\infty;\rho)$.

(d) Conversely, if a compact set $K \subset \mathbb{C}$ is given, then there is $\rho > 0$ such that $B_\infty(\infty;\rho) \subset \mathbb{C}_\infty \setminus K$.

Note. The following result shows that metrics $|\cdot|$ and d determined the same open sets on $\mathbb{C}$. That is, $|\cdot|$ and d induce the same (metric) topology on $\mathbb{C}$. This is spelled out in the following result.

Topologies on $\mathbb{C}_\infty$ Theorem. Let $G \subset \mathbb{C}$. Then G is open in metric space $(\mathbb{C}, |\cdot|)$ if and only if G is open in metric space $(\mathbb{C}_\infty, d)$.

Definition II.3.1. If $\{x_1, x_2, \ldots\}$ is a sequence in a metric space (X, d) then $\{x_n\}$ *converges* to x, denoted $x = \lim x_n$ or $x_n \to x$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x, x_n) < \varepsilon$.

Note. When two different metrics induce the same topology on a set, they imply the same convergence of sequences. That is, we have the following theorem.

Sequences in $\mathbb{C}_\infty$ Theorem. Let $\{z_n\}_{n=1}^\infty$ be a sequence of complex numbers. Then for $z \in \mathbb{C}$, $z_n \to z$ in metric space $(\mathbb{C}, |\cdot|)$ if and only if $z_n \to z$ in metric space $(\mathbb{C}_\infty, d)$.

Definition II.3.5. A sequence $\{x_n\}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. If metric space (X, d) has the property that each Cauchy sequence has a limit in X, then (X, d) is *complete*.

Note. Exercise II.3.4 states: Let $a_n, z \in \mathbb{C}$ and let d be the metric on $\mathbb{C}_\infty$. Then $|z_n - z| \to 0$ if and only if $d(z_n, z) \to 0$. Also, if $|z_n| \to \infty$ then $\{z_n\}$ is a Cauchy sequence in $\mathbb{C}_\infty$.

Note. In Introduction to Topology (MATH 4357/5357), we introduce the idea of compactification. From Section 3.29, "Local Compactness," in Chapter 3, "Connectedness and Compactness," of James Munkres' *Topology*, 2nd edition (Prentice Hall, 2000) we have the following.

> **Definition.** If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is a *compactification* of X. If $Y \setminus X$ is a single point, then Y is the *one-point compactification* of X.

All metric spaces are "Hausdorff," so this doesn't affect us. For more information, see my online notes for Introduction to Topology.

Compactness of $\mathbb{C}_\infty$ Theorem.

$\mathbb{C}_\infty$ is a compact metric space under d.

Note. Corollary II.4.5 state that "Every compact metric space is complete." Therefore the Compactness of $\mathbb{C}_\infty$ Theorem gives that $\mathbb{C}_\infty$ is also complete (that is, Cauchy sequences converge).

Note. We see that $\mathbb{C}_\infty$ is the one-point compactification of $\mathbb{C}$. We can similarly define the one-point compatification of $\mathbb{R}$ as $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ (this is to be dealt with in Conway's Exercise II.3.7).

Note. In Section V.3, "The Argument Principle," we define a function on open set $G \subset \mathbb{C}$ as a *meromorphic* function on G if f is analytic on G, except for poles. In Section V.1, point $z = a$ is defined as a *pole* of function f if a is an isolated singularity of f and $\lim_{z \to a} |f(z)| = \infty$.

Note. If f is meromorphic on open set G and we define $f(z) = \infty$ at each pole of meromorphic function f, then by Exercise V.3.4, $f : G \to \mathbb{C}_\infty$ is continuous function from metric space $(\mathbb{C}, |\cdot|)$ to metric space $(\mathbb{C}_\infty, d)$.

Note. In analysis, it is common to study "spaces" of functions. In Chapter VII, "Compactness and Convergence in the Spaces of Analytic Functions," we will consider spaces of continuous functions, analytic functions, and meromorphic functions. Completeness of these spaces is a common topic. The following result from Chapter VII shows that the meromorphic functions, along with the function $f : \mathbb{C} \to \mathbb{C}_\infty$ which takes on the constant value ∞, is a complete metric space.

Corollary VII.3.5. Let G be a region in $\mathbb{C}$ (i.e., an open connected subset in $\mathbb{C}$). Then the meromorphic functions on G combined with $f \equiv \infty$ on G, $M(G) \cup \{\infty\}$, is a complete metric space under metric ρ.

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