Chapter I. The Complex Number System

I.1. The Real Numbers

Note. In this section, we define the real numbers as you do in Analysis 1 (MATH 3127/5127). This requires a review of the definition of a "field." We also give a bit of history on the introduction of complex numbers.

Note 1.1.A. The following definitions can be found in my online notes for Introduction to Modern Algebra (MATH 4127/5127):

Definition 18.1. A *ring* $\langle R, +, \cdot \rangle$ is a set R together with two binary operations $+$ and $\cdot$, called *addition* and *multiplication*, respectively, defined on R such that:

$\mathcal{R}_1$**:** $\langle R, + \rangle$ is an abelian group.

$\mathcal{R}_2$**:** Multiplication $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.

$\mathcal{R}_3$**:** For all $a, b, c \in R$, the *left distribution law* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and the *right distribution law* $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ hold.

Definition 18.14. A ring in which multiplication is commutative (i.e., $ab = ba$ for all $a, b \in R$) is a *commutative ring*. A ring with a multiplicative identity element is a *ring with unity*.

Definition 18.16. Let R be a ring with unity $1 \neq 0$. An element $u \in R$ is a *unit* of R if it has a multiplicative inverse in R. If every nonzero element of R is a unit, then R is a *division ring*. A *field* is a commutative division ring.

Note 1.1.B. We now shift to Analysis 1 (MATH 4217/5217) for more definitions. See my online notes for Analysis 1 on Section 1.2. Properties of the Real Numbers as an Ordered Field (which gives a more direct definition of a *field*) and Section 1.3. The Completeness Axiom.

Definition. An *ordering* of a field $\mathbb{F}$ is a set P such that

1. P is closed under addition,
2. P is closed under multiplication, and
3. P satisfies the Law of Trichotomy: For each $f \in \mathbb{F}$, exactly one of the following holds $f \in P$, $-f \in P$, or $f = 0$.

For $a, b \in \mathbb{F}$, if $b - a \in P$, then we say $a < b$.

Definition. Let A be a subset of an ordered field $\mathbb{F}$. If there exists $b \in \mathbb{F}$ such that for all $a \in A$, then b is an *upper bound* of A and A is said to be *bounded above*. If there is a $c \in \mathbb{F}$ such that $c \leq a$ for all $a \in A$, then c is a *lower bound* of A and A is *bounded below*. A set bounded above and below is *bounded*. A set that is not bounded is *unbounded*.

Definition. Let A be a subset of an ordered field $\mathbb{F}$ which is bounded above. Then $b \in \mathbb{F}$ is called a *least upper bound* (*lub* or *supremum*) of a set A if (1) b is an upper bound of A and (2) if c is an upper bound of A, then $b \leq c$.

Definition. An ordered field is *complete* if every set with an upper bound has a least upper bound.

Note 1.1.C. We now have the equipment to define the real numbers.

Definition. The *real numbers*, denoted $\mathbb{R}$, are a complete ordered field.

Note 1.1.D. We will see in Section I.2. The Field of Complex Numbers that the complex numbers, in fact, form a field. However, we'll see in Supplement. Ordering the Complex Numbers that the complex numbers do not form an ordered field.

Note 1.1.E. It is natural to ask if there are multiple complete ordered fields, with $\mathbb{R}$ as just one such example. This is addressed in Michael Henle's *Which Numbers are Real?* (Mathematical Association of America, 2012). See Henle's Section 2.3 "Uniqueness of the Reals." In particular, notice the following on page 48:

Theorem 2.3.3. Every complete, ordered field is isomorphic to $\mathbb{R}$.

So the real numbers are the *unique* complete ordered field (well, up to isomorphism).

Note. We will need to address completeness in the field of complex numbers, $\mathbb{C}$. Since the complex numbers form a field but do not have an ordering (see Note 1.1.D above), we will need to address completeness in some other way. We do so in Section I.3. The Complex Plane (see Note 1.3.D). Remember that completeness is what makes a structure a continuum. So if we are doing analysis, then we *must have completeness*; this guarantees that the limits that *should* exist (namely, those of Cauchy sequences), *do* exist.

Note 1.1.F. Complex numbers enter mathematics with the establishment of the theory of equations in the 16th century (in fact, negative numbers gain a larger acceptance in this setting as well). Girolamo Cardano (September 24, 1501–September 21, 1576) in his *Ars Magna* ("The Great Art") of 1545 stated the following problem: "Divide 10 into two parts whose product is 40." Using the quadratic formula, it is easy to see that we need $5-\sqrt{-15}$ and $5+\sqrt{-15}$. Certainly these add to 10 and their product is 40. However, to take the product we must deal with the complex numbers $\pm\sqrt{15}i$ and their arithmetic. Cardano states that "putting aside the mental tortures involved," this is the solution. Cardano also stated the solution to cubic equations with a sort-of "cubic formula" for third degree polynomial equations, similar to the quadratic formula for second degree polynomial equations. The cubic formula was due to Niccolò Tartaglia (1500–December 13, 1557) and Cardano published it without Tartaglia's permission, leading to a famous mathematical argument between the two. More on this can be found in my online notes for Introduction to Modern Algebra (MATH 4127/5127) on Supplement. Why The Hell Am I In This Class? (a motivational lecture on how modern algebra is related to classical algebra) and in my online notes for History of Mathematics (MATH 3040) on Section 8.8. Cubic and Quartic Equations. The relevance of the cubic formula to the history of complex numbers is that square roots of negative real numbers appear in the formula in certain cases. For example, if we consider $x^3-15x-4=0$, the cubic formula gives as a solution $x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$. This value of x can be manipulated with the usual algebraic rules to show that it simplifies to $x=4$, which is clearly a solution; these manipulations again requires Cardano's "putting aside the mental tortures involved." What we see is that the usual manipulation of the symbol $\sqrt{-121}$ (such as $\sqrt{-121}\sqrt{-121}=(\sqrt{-121})^2=-121$ and $\sqrt{-121}-\sqrt{-121}=0$) ultimately leads to the mental-torture-free solution of 4. This showed that square roots of negative numbers can prove useful in algebraic manipulations. This usefulness in the theory of equations would lead ultimately to the acceptance of them as actual numbers (and similarly for negative numbers).

 

Girolamo Cardano (left) and Niccolò Tartaglia (right)

The images above and this note are based on the MacTutor biography webpages on Cardano and Tartaglia (accessed 8/21/2023).

Note 1.1.G. The complex numbers were not an immediate success. Some supported the complex solutions, such as French mathematician and musician Albert Girard (1595–December 8, 1632; he introduced the abbreviations sin, cos, and tan for trigonometric functions). Others opposed complex solutions, such as Italian mathematician Rafael Bombelli (January 1526–1572; he published his influential *Algebra* in 1572 in three volumes). Probably the most famous in opposition to complex numbers was René Descartes (March 31, 1596–February 11, 1650). In his *La Géométrie*, in which he introduced analytic geometry and the Cartesian plane, he described square roots of negatives as: "Neither the true nor the false [negative] roots are always real; sometimes they are *imaginary*." Descartes did not recognize imaginary numbers as numbers (and hence he did not recognize complex numbers, that is sums of real numbers and imaginary numbers, as numbers). Of course the term "imaginary" is still with us. . . regrettably.

Rene Decartes

This note (and the image of Descartes above) is based on the MacTutor biography webpages on Albert Girard, Rafael Bombelli, and René Descartes (accessed 8/21/2023). For more on Tartaglia, Cardano, and Bombelli, see my online notes for History of Mathematics (MATH 3040) on Section 8.8. Cubic and Quartic Equations. For more on Descartes and Girard, see my online notes for the same class on Section 10.2. Descartes and Section 10.7. Some Seventeenth-Century Mathematicians of Germany and the Low Countries, respectively.

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