I.2. The Field of Complex Numbers

Note 1.2.A. We give a definition of the complex numbers based on the "Cayley-Dickson Construction" which allows us to make a 2-dimensional *algebra* $\mathbb{C}$ using the 1-dimensional algebra $\mathbb{R}$ (an *algebra* A is a vector space, over $\mathbb{R}$ here, that is equipped with a bilinear map [that is, linear in both entries] of *multiplication* mapping $A \times A \to A$). The Cayley-Dickson Construction can be used to create the 4-dimensional (noncommutative) algebra of the quaternions $\mathbb{H}$, and then used to create the 8-dimensional (noncommutative and nonassociative) algebra of the octonions $\mathbb{O}$. The construction can be iterated to produce more and more algebras (all of dimension 2^n over $\mathbb{R}$, for some $n \in \mathbb{N}$), though the algebras loose structure after we go past the octonions (or, maybe even, after we go past $\mathbb{C}$). More details on this are in my supplemental online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Supplement. The Cayley-Dickson Construction and Nonassociative Algebras.

Note 1.2.B. As an early example of the Cayley-Dickson Construction, we observe that William Rowan Hamilton (August 4, 1805–September 2, 1865) presented the paper "Theory of Conjugate Functions, or Algebraic Couples: with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time" to the Royal Irish Academy on November 4, 1833. It appeared in print in *The Transactions of the Royal Irish Academy*, **17**, 293–423 (1831). You possibly know Hamilton from his introduction of the quaternions in 1843 (ten years after his presentation on complex numbers). He defined a complex number as a pair of real numbers, $z = (x, y)$, and the defined addition and multiplication as

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$\text{and } z_1 z_2 = (x_1, y_1)(x_2 y_2) = (x_1 x_2 - y_1 y_2, x_2 y_1 + x_1 y_2).$$

In addition, the conjugate of $z = (x, y)$ is $\overline{z} = (x, y)^* = (x, -y)$ and the modulus of $z = (x, y)$ is $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$. This is the approach we take in defining the field of complex numbers. In fact, Hamilton *could have used* a similar definition of quaternions as ordered pairs of complex numbers! He did not, but Arthur Cayley (August 16, 1821–January 26, 1895) discovered that he could take ordered pairs of quaternions and construct a real algebra (today, called a Cayley-Dickson algebra) of dimension eight called the octonions in 1845 (though John Graves [December 4, 1806–March 29, 1870] found them earlier in 1843). Details are given in Supplement. The Cayley-Dickson Construction and Nonassociative Algebras mentioned in Note 1.2.A. For our purposes, we only need the first application of the Cayley-Dickson Construction to get the complex field from the real field.

Definition. The set of *complex numbers* is $\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$. Define *addition* on $\mathbb{C}$ as $(a, b) + (c, d) = (a + c, b + d)$ and *multiplication* on $\mathbb{C}$ as $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$.

Note 1.2.C. One with too much time on their hands can verify that $\langle \mathbb{C}, +, \cdot \rangle$ is a field. The additive identity is $(0, 0)$ and the multiplicative identity is $(1, 0)$. The additive inverse of (a, b) is $(-a, -b)$ and the multiplicative inverse of $(a, b) \neq (0, 0)$ is $\left(\dfrac{a}{a^2 + b^2}, \dfrac{-b}{a^2 + b^2} \right)$.

Note. From now on, we denote (a, b) as $a + ib$. Notice then that $i^2 = -1$. With $z = a + ib$, we call a the *real part* of z, $\mathrm{Re}(z)$, and b the *imaginary part* of z, $\mathrm{Im}(z)$.

Note 1.2.D. We now consider an imagined dialogue on the relationship between $\mathbb{C}$ and $\mathbb{R}^2$.

Question. Is $\mathbb{C}$ isomorphic to $\mathbb{R}^2$?

Answer. As a what? A field?

Question (revised). Is the field $\mathbb{C}$ isomorphic to the field $\mathbb{R}^2$?

Answer. NO! $\mathbb{R}^2$ is not a field, it's a vector space!

Question (re-revised). Is the vector space $\mathbb{C}$ isomorphic to the vector space $\mathbb{R}^2$?

Answer. That's a good question! However, it is meaningless/misleading. A vector space isomorphism is only defined between two vector spaces over the same field. $\mathbb{R}^2$ is a two dimensional field over $\mathbb{R}$ and $\mathbb{C}$ is a one dimensional vector space over field $\mathbb{C}$. However, $\mathbb{C}$ can be viewed as an *extension field* of $\mathbb{R}$ which treats $\mathbb{C}$ as a two dimensional vector space over $\mathbb{R}$ with basis $\{1, i\}$. So we can legitimately state that: "$\mathbb{C}$ is a two dimensional extension field of $\mathbb{R}$"; see my online notes for Modern Algebra 2 (MATH 5420) in Section V.1. Field Extensions (see Example V.1.A). We can also legitimately state that: "$\mathbb{C}$ is a two dimensional *algebra* over $\mathbb{R}$," as described in Note 1.2.A.

If you have a familiarity with extension fields, then you know that there are a bunch of fields between $\mathbb{Q}$ and $\mathbb{R}$ (such as $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, the algebraic numbers, etc.; see, for example, my online notes for Introduction to Modern Algebra 2 [MATH

4137/5137] on Section VI.31. Algebraic Extensions). We might wonder if there is similarly several fields $\mathbb{R}$ and $\mathbb{C}$. However, this is not the case, as is shown in Modern Algebra 2 (MATH 5420); see my online notes for that class on Section IV.2. Free Modules and Vector Spaces and notice Note IV.2.K and Exercise IV.2.6(b) in that note.

Definition. For $z = a + ib \in \mathbb{C}$, the *absolute value* (or *modulus*) of z is $|z| = \sqrt{a^2 + b^2}$. The *conjugate* of $z = a + ib$ is $\overline{z} = a - ib$.

Notice 1.2.E. $|z|^2 = z\overline{z}$ and $z^{-1} = \overline{z}/|z|^2$. Notice that this gives $\overline{z^{-1}} = \overline{1/z} = \overline{\overline{z}/|z|^2} = z/|z|^2$. These are equation (2.1) in the text.

Theorem I.2.A. For all $z, w \in \mathbb{C}$ we have:

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z}) \tag{2.2}$$

$$\overline{z + w} = \overline{z} + \overline{w} \qquad \overline{zw} = \overline{z}\,\overline{w} \tag{2.3}$$

$$|zw| = |z||w| \tag{2.4}$$

$$|z/w| = |z|/|w| \tag{2.5}$$

$$|\overline{z}| = |z|. \tag{2.6}$$

Note. A proof of Theorem I.2.A is to be given in Exercise I.2.5.

Corollary I.2.A. For all $z, w \in \mathbb{C}$ we have $\overline{z/w} = \overline{z}/\overline{w}$.

Proof. We have

$$\begin{aligned}\overline{z/w} &= \overline{z(1/w)} = \overline{z}\,\overline{1/w} \text{ by equation (2.3)} \\ &= \overline{z}w/|w|^2 \text{ by equation (2.1)} \\ &= \overline{z}\frac{w}{w\overline{w}} = \overline{z}/\overline{w}.\end{aligned}$$

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