I.3. The Complex Plane

Note 1.3.A. Caspar Wessel (June 8, 1745–March 25, 1818) was a Norwegian mathematician who first gave a geometric interpretation to the representation of complex numbers. He attended Christiania Cathedral School in Christiania, Norway (which is modern day Oslo). This is the same school later attended by the algebraist Ludwig Sylow (December 12, 1832–September 7, 1918), of "Sylow's Theorems" fame. He studied law at the University of Copenhagen, and assisted his bother is surveying projects. This lead to additional work as a surveyor and map creator. During this work, he developed his mathematical skills. His mathematical fame is based on his only mathematical paper. On March 10, 1797 he presented the paper, "On the Analytical Representation of Direction; An Attempt, Applied Chiefly to the Solution of Plane and Spherical Polygons," to the Royal Danish Academy. It was published (in Danish) in 1799. In this paper, he introduced the idea of the "complex plane" as described in this section. Ironically, his geometric interpretation is usually called the "Argand diagram." This is because Wessel's paper was not well-known in the mathematical community until 1895 when Danish geometer Christian Juel (January 25, 1855–January 24, 1935) called attention to it. However, by then Swiss mathematician Jean R. Argand (July 18, 1768–August 13, 1822) had been recognized for his introduction of this idea in 1806 (Carl F. Gauss [April 30, 1777–February 23, 1855] also independently introduced the idea in 1831). Wessel's original paper was not available in English until 1929 when it appears in David Smith's *A Source Book in Mathematics* (London, 1929). The English translation was by Martin Nordgaard and appeared in the book on pages

55 to 66 as "Wessel on Complex Numbers." The information for this note (and the image below) is from the MacTutor biography webpage of Caspar Wessel (accessed 8/21/2023).

Note 1.3.B. Jean-Robert Argand (July 18, 1768–August 13, 1822) was an amateur Swiss mathematician. The biographical information on Argand is sketchy (including his first name, and birth and death dates). He was an accountant and bookkeeper in Paris. Argand sent a copy of his work "Essay on a Way of Representing Imaginary Quantities in Geometric Constructions" to Adrien-Marie Legendre (September 18, 1752–January 10, 1833). Legendre passed the letter on to François Français (April 7, 1768–October 20, 1810) on November 2, 1806. After François Français's death four years later, his brother Jacques Français (June 20, 1775–March 9, 1833) went through his papers and found the letter. In September 1813 Jacques Français published "New Principles of Position Geometry, and Interpretation of Imaginary Symbols" in *Annales de Mathématiques*. This lead to some controversy which played out in *Annales de Mathématiques*, which likely

helped attach Argand's name to the geometric interpretation of complex numbers in an "Argand diagram." Argand published eight more papers, all in *Annales de Mathématiques*, between 1813 and 1816. In the last one, he introduced the notation (m, n) for the number of combinations of n objects selected from a set of m objects. This history is base on the MacTutor biography webpage of Jean-Robert Argand (accessed 8/21/2023).

Note 1.3.C. We have a "natural relationship" between $\mathbb{R}^2$ and $\mathbb{C}$, so we use this for a geometric interpretation of $\mathbb{C}$. We introduce a "real axis" and "imaginary axis," then plot $z = a + ib$ in the resulting Cartesian plane. We then see that addition in $\mathbb{C}$ corresponds to vector addition in $\mathbb{R}^2$:
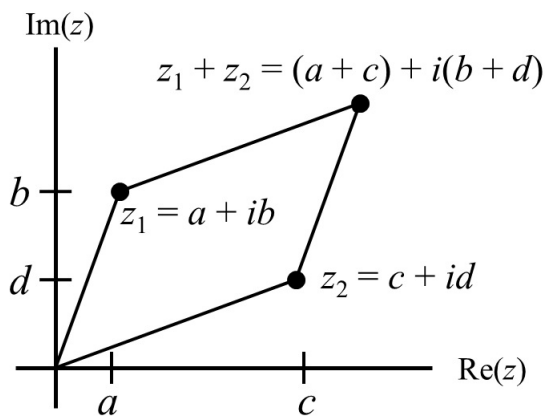
This is an example of an Argand diagram. That is, the addition of complex numbers satisfies the Parallelogram Law of Addition:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \quad \text{(Exercise I.2.4c)}.$$

"The sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of its two diagonals."

Note. We now show that modulus satisfies a triangle inequality and hence the modulus defines a *norm* on $\mathbb{C}$ (where $\mathbb{C}$ is treated as a vector space). Modulus also defines a *metric* on $\mathbb{C}$, and $\mathbb{C}$ can be treated as a metric space, with a resulting topology. This is the topic of our Chapter II, "Metric Spaces and the Topology of $\mathbb{C}$." We start by establishing the Triangle Inequality for the norm defined by the modulus of a complex number.

Theorem I.3.A. The Triangle Inequality.

For all $z, w \in \mathbb{C}$, $|z + w| \leq |z| + |w|$.

Note. Whenever we state an inequality, it is of interest to determine when it reduces to an equality. Just as in a real vector space where we have $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$, for nonzero vectors $\vec{u}$ and $\vec{v}$, if and only if $\vec{u} = t\vec{v}$ for some $t \geq 0$, we have equality in Theorem I.3.A under similar conditions, as follows.

Corollary I.3.A. For nonzero $z, w \in \mathbb{C}$, $|z + w| = |z| + |w|$ if and only if $z = tw$ for some $t \in \mathbb{R}$, $t \geq 0$.

Note. Another useful inequality, based on the Triangle Inequality, is: For all $z, w \in \mathbb{C}$, $||z| - |w|| \leq |z - w|$. This is Exercise I.3.1.

Note. We can use the modulus as a metric on $\mathbb{C}$ and so we can discuss sequences, convergence, and the property of "Cauchyness." We can address completeness of $\mathbb{C}$ using Cauchy sequences with the following. You will see in its proof that the result ultimately follows from the completeness of the real numbers (that is, from the fact that a Cauchy sequence of real numbers converges; see Note 1.3.D below).

Theorem I.3.B. Cauchy Sequences Theorem. A Cauchy sequence of complex numbers is convergent.

Note. With the Triangle Inequality, it is easy to see that a convergent sequence of complex numbers is Cauchy. Therefore, a sequence of complex numbers is Cauchy if and only if it is convergent.

Note 1.3.D. In many settings (metric spaces) where there is no ordering (such as $\mathbb{C}$, $\mathbb{R}^n$ for $n \geq 2$, $\mathbb{C}^n$, and ℓ^2), the definition of "complete" is that Cauchy sequences converge. In the setting of $\mathbb{R}$, we could replace the Axiom of Completeness (sets of real numbers with upper bounds have a least upper bound) with the assumption that Cauchy sequences converge, and we would get the same structure for $\mathbb{R}$. That is, these statements are equivalent. Therefore, the above results show that $\mathbb{C}$ is complete under the alternate definition.

Note 1.3.E. By the way, Augustin Louis Cauchy (August 21, 1789–May 23, 1857) *assumed* "Cauchy sequences" converge (circa 1820). Bolzano used this assumption

to prove that a set of real numbers which is bounded above has a least upper bound (1817). Richard Dedekind introduced the idea of a "Dedekind cut" (in 1858, but not published until 1872) as an Axiom and it is from this point that the real numbers were clearly defined axiomatically. See Jacqueline Stedall's *Mathematics Emerging: A Sourcebook 1540–1900* (Oxford University Press, 2008), for more history (notice Section 16.1, "Cauchy Sequences").

Image of Augustin Cauchy from the MacTutor biography webpage of Cauchy (accessed 8/21/2023)

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