

I.4. Polar Representation and Roots of Complex Numbers

Note. Since an element of \mathbb{R}^2 can be represented in polar coordinates (r, θ) , then there is a similar representation of elements of \mathbb{C} . In this section we give some history of polar coordinates and their use to represent complex numbers. We state De Moivre's Formula and use it to calculate roots of a complex number.

Note 1.4.A. The history of polar coordinates in the Cartesian plane seems murky. Bonaventura Cavalieri (1598–November 30, 1647) was an Italian mathematician known mostly for his contribution to integral calculus and the method of indivisibles (this is addressed in my online notes for History of Mathematics [MATH 3040] on [Section 11.6. Cavalieri's Method of Indivisibles](#)). In 1635 he published *Geometrica indivisibilibus continuorum* (“Geometric Indivisible Continua”) (with a second, corrected edition published in 1653). His concern is on the area within a curve (and hence his contributions to integration), in particular the area within an Archimedian spiral which he related to the area outside of a parabola. Gregorius Saint-Vincent (September 8, 1584–January 27, 1667) published *Opus Geometricum* (“Geometrical Work”) in 1647 in which he claimed to have already known the method used by Cavalieri, setting up a priority controversy. This introductory work was done with an eye towards computation of areas using transformations involving polar coordinates, and not in terms of graphs of curves in a new coordinate system. The first to use polar coordinates to graph and visualize points and curves is Isaac Newton (January 4, 1643–March 31, 1727; under the new calendar) in his *The Method of Fluxions* (London, 1736). However, Newton used polar coordinates in special settings and to specific ends. The person usually stated as

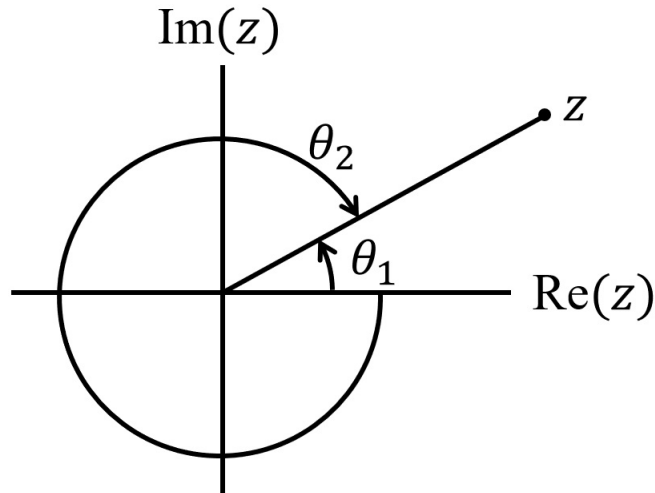
the “discoverer” or “inventor” of polar coordinates is Jacob Bernoulli (January 6, 1655–August 16, 1705). He uses polar coordinates to locate any point in the plane. This is given in his “Spécimen calculi differentialis in dimensione parabolae helicoidis,” (“A model of differential calculus in the dimension of a helicoidal parabola”), *Acta Eruditorum* (1691). Because of his generalized use of them in this work, you may find statements along the lines that: “Jacob Bernoulli invents polar coordinates [in 1691], a method of describing the location of points in space using angles and distances.” (See, for example, [MacTutor webpage on “Chronology 1675–1700”](#), accessed 9/3/2023.) The main source for this note is J. L. Coolidge’s “The Origin of Polar Coordinates,” *The American Mathematical Monthly*, **59**(2), 78–85 (February, 1952); this can be viewed online on [JSTOR](#) (accessed 9/3/2023).



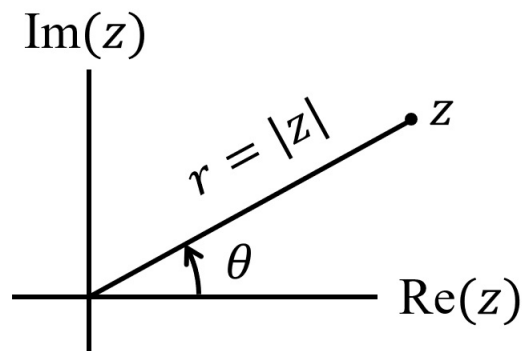
Images from the MacTutor biography webpages on [Cavalieri](#) (left), [Newton](#) (center), and [Jacob Bernoulli](#) (right); each accessed 9/3/2023.

Next, we use polar coordinates to give a representation of complex numbers. We know a complex number can be associated with a point in the Cartesian plane, the Argand diagram described in Note 1.3.C of [Section I.3. The Complex Plane](#).

Definition. Let $z \in \mathbb{C}$ and let θ be an angle between the positive real axis and the line joining 0 and z ($z \neq 0$). Then θ is an *argument* of z , denoted $\theta = \arg(z)$. (We cannot think of $\arg(z)$ as a function—it is best to think of it as a set.)



Note. If $|z| = r$ and $\theta = \arg(z)$, then we have:



Then $\operatorname{Re}(z) = r \cos \theta$ and $\operatorname{Im}(z) = r \sin \theta$. Therefore, $z = r(\cos \theta + i \sin \theta)$. We denote $\cos \theta + i \sin \theta$ as $\operatorname{cis}(\theta)$.

Note. The next result follows by “the formulas for the sine and cosine of the sum of two angles,” as Conway states it on page 5 of the textbook. You can find

a proof in my online notes for Complex Variables (MATH 4337/5337) on [Section 1.7. Products and Powers in Exponential Form](#) (see Theorem 1.7.1; exponential notation is used, but computations are done using sines and cosines as we would need here).

Theorem I.4.A. Suppose $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$ where $r_k = |z_k|$ and $\theta_k = \arg(z_k)$ for $k = 1, 2$. Then $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$. That is, “ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.”

Note. By applying mathematical induction, we get the following as a corollary to Theorem I.4.A.

Corollary I.4.A. de Moivre’s Formula.

If $z = r \operatorname{cis}(\theta)$, then $z^n = r^n \operatorname{cis}(n\theta)$. In particular, $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$.

Note 1.4.B. Abraham de Moivre (May 26, 1667–November 27, 1754) is known for his pioneering work in analytic geometry and probability theory. We give as a very brief biography of him, the following from Paul Nahin’s *An Imaginary Tale: The Story of $\sqrt{-1}$* , Princeton University Press (1998), pages 56 and 57 (this is a nice, surprisingly technical, history of complex analysis):

“... ‘De Moivre’s theorem,’ after the French-born mathematician Abraham De Moivre (1667–1754). De Moivre, a Protestant, left Catholic France at age eighteen to seek religious freedom in London, where he became a friend of Isaac Newton. In a 1698 paper published in the *Philosophical Transactions* of the Royal Society, he mentions that Newton knew of an equivalent expression of De Moivre’s theorem as early as 1676, which Newton used to calculate the cube roots of the complex numbers that come out of the Cardan formula for the irreducible case. ... It is clear from De Moivre’s writings that he did, in fact, know and use the above result, but he never actually wrote it out explicitly—that was done by Euler in 1748, who arrived at it by entirely different means. ...”

It is not unusual for mathematical results to be arrived at independently. It is a little unusual for the correct credit to be given (we have not seen the last of Euler, and he already has enough results to his name anyhow!).

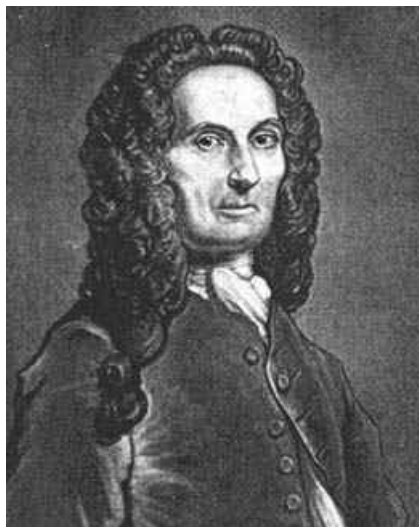


Image from the [MacTutor biography webpage for de Moivre](#) (accessed 9/3/2023).

Note. We can use de Moivre's Formula to compute roots. Let $w \in \mathbb{C}$, $w \neq 0$. We want to find all $z \in \mathbb{C}$ such that $z^n = w$ (for a given $n \in \mathbb{N}$). For such a z , we need $|z| = |w|^{1/n}$ and $\arg(z) = \arg(w)/n$. Let $\alpha = \arg(w)$, then one such z is $z = |w|^{1/n} \text{cis}(\alpha/n)$. However, there are several choices for α . We find that there are n such z and are given by

$$|w|^{1/n} \text{cis} \left(\frac{\alpha + 2k\pi}{n} \right) \text{ for } k = 0, 1, \dots, n-1 \quad (4.5)$$

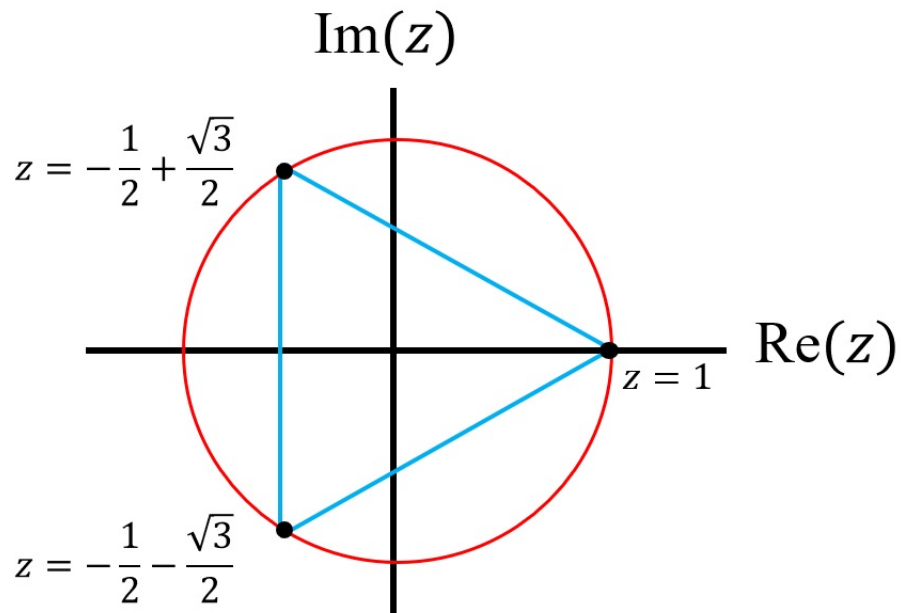
(where α is any argument of z). These roots are distributed uniformly around a circle centered at the origin of the complex plane with radius $|w|^{1/n}$, starting at $|w|^{1/n} \text{cis}(\alpha/n)$. We now illustrate this.

Example. Find three cube roots of 1 and graph them in the complex plane.

Solution. We have $w = 1$ and $n = 3$. Since 1 is a positive real number, we can take $\alpha = 0$. From equation (4.5), the desired roots are $|1|^{1/3} \text{cis}((0+2k\pi)/3) = \text{cis}(2k\pi/3)$ for $k = 0, 1, 2$. So the desired roots are:

$$\begin{aligned} \text{cis}(0) &= \cos 0 + i \sin 0 = 1, \\ \text{cis}(2\pi/3) &= \cos(2\pi/3) + i \sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \\ \text{cis}(4\pi/3) &= \cos(4\pi/3) + i \sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

So the roots are distributed along the vertices of an equilateral triangle, as follows.



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