I.5. Lines and Half-Planes in $\mathbb{C}$

Note. A line in $\mathbb{C}$ is a collection of complex numbers which give a line when plotted in an Argand diagram. A half-plane in $\mathbb{C}$ is the collection of complex numbers on one side of a line in $\mathbb{C}$. In this brief section we give an equation for a line and a half-plane in $\mathbb{C}$. We'll see that this involves a condition on the imaginary part of a quantity involving variable complex number z.

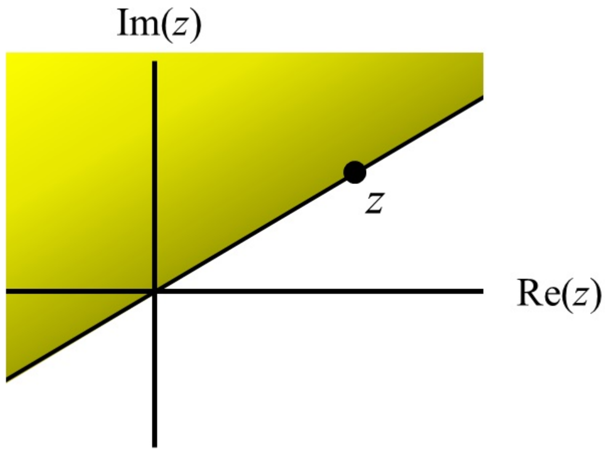
Note. In Calculus 3 (MATH 2110) we see that the parametric equation for a line in $\mathbb{R}^2$ is $(x, y) = (a_1, a_2) + t(b_1, b_2)$ where $t \in \mathbb{R}$ (see my online Calculus 3 notes on Section 12.5. Lines and Planes in Space). The line passes through the point (a_1, a_2) (when $t = 0$) and has direction vector $[b_1, b_2]$ (where we notationally distinguish between the coordinates of a point (x, y) and the components of a vector $\vec{u} = [x, y]$). So the equation of a line in $\mathbb{C}$ is $z = a + tb$ where $t \in \mathbb{R}$ and $a, b \in \mathbb{C}$. This can be rearranged as $t = (z - a)/b$. Since t is real, the equation of a line in $\mathbb{C}$ is of the form

$$\mathrm{Im}\left(\frac{z-a}{b}\right) = 0 \text{ where } b \neq 0.$$

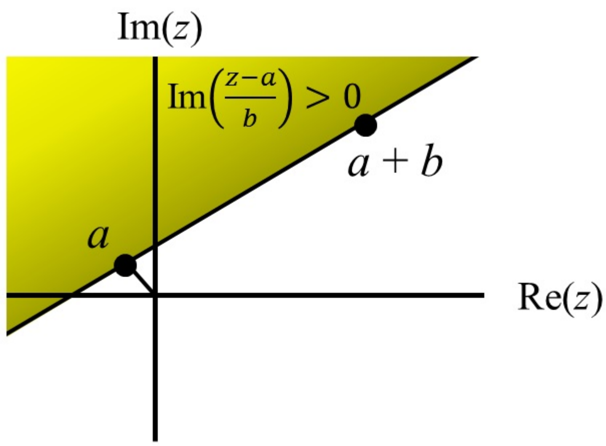
Note. Let $a = 0$ and $b = \mathrm{cis}(\beta) = \cos\beta + i\sin\beta$ (i.e., without loss of generality $|b| = 1$). Then the line is $\mathrm{Im}(z/b) = 0$. If $z = r\mathrm{cis}(\theta)$ and $\mathrm{Im}(z/b) > 0$, then

$$\mathrm{Im}(r\mathrm{cis}(\theta - \beta)) = \mathrm{Im}(r(\cos(\theta - \beta) + i\sin(\theta - \beta)) = r\sin(\theta - \beta) > 0$$

and so $r\sin(\theta - \beta) > 0$. Now $\sin(\theta - \beta) > 0$ is satisfied when $0 < \theta - \beta < \pi$, or $\beta < \theta < \beta + \pi$. So $\mathrm{Im}(z/b) > 0$ is the half plane:



If we translate all z satisfying $\text{Im}(z/b) > 0$ by an amount a, we get the half plane:



Notice that if we interpret b as a vector and we travel along the line in the direction b, then $\text{Im}\left(\frac{z-a}{b}\right) > 0$ lies to the left of the line and $\text{Im}\left(\frac{z-a}{b}\right) < 0$ lies to the right of the line.

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