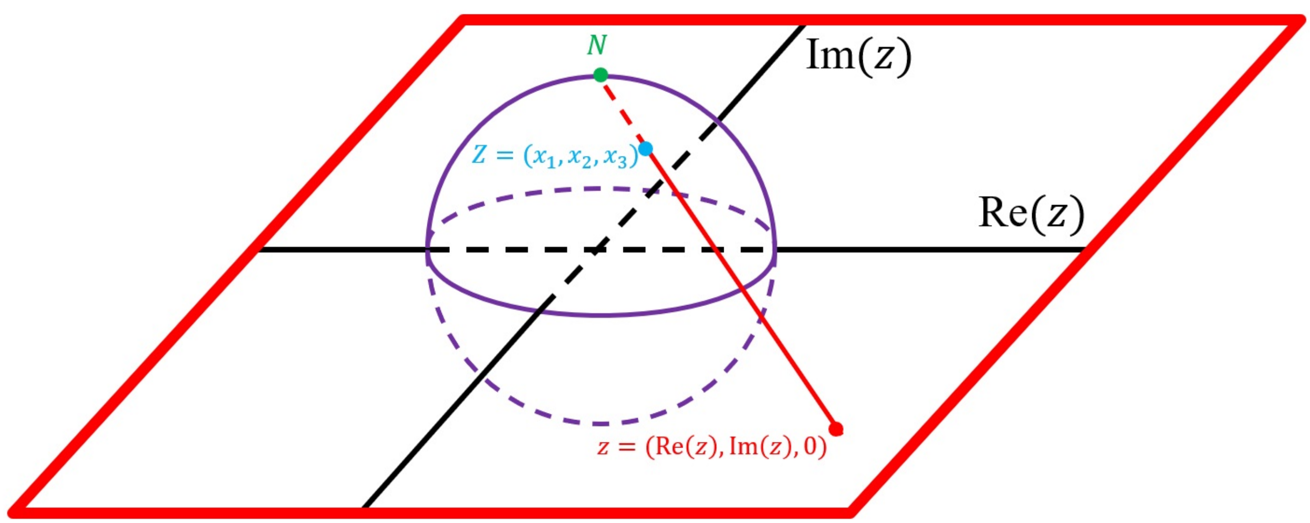
I.6. The Extended Plane and Its Spherical Representation

Note. In this section we introduce a way of measuring distance (i.e., a *metric*) on the *extended complex plane* $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The fact that our distance function *is* a metric will be considered in Supplement. The Extended Complex Plane (in fact, it will be shown in this supplement that the extended complex plane is a compact set). In Section VII.3. Spaces of Meromorphic Functions, we consider functions with isolated poles (f has a pole at $z = a$ if $\lim_{z\to a} |f(z)| = \infty$) and allow these functions to take on the value ∞, so that meromorphic functions map $\mathbb{C}$ to $\mathbb{C}_\infty$. We will see the distance function from this section again there.

Note. We define a one-to-one and onto function from the unit sphere in $\mathbb{R}^3$ to $\mathbb{C}_\infty$. We'll use this projection to define distance in $\mathbb{C}_\infty$:



Note. Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$; this is called the *Riemann sphere*. We map the north pole N to infinity and we project $Z = (x_1, x_2, x_3) \in S$, $Z \neq N$, to the point $z = (\mathrm{Re}(z), \mathrm{Im}(z), 0)$ where $z = \mathrm{Re}(z) + i\mathrm{Im}(z) \in \mathbb{C}$, by projecting Z onto $\mathbb{C}$ with a line through N and Z with the axes laid out as above.

Note. With the techniques of Calculus 3 (MATH 2110), see in particular Section 12.5. Lines and Planes in Space, we can show that: $x_1 = \dfrac{z + \overline{z}}{|z|^2 + 1}$, $x_2 = \dfrac{-i(z - \overline{z})}{|z|^2 + 1}$, $x_3 = \dfrac{|z|^2 - 1}{|z|^2 + 1}$, and $z = \dfrac{x_1 - ix_2}{1 - x_3}$.

Definition. The above relations between S and $\mathbb{C}_\infty$ are called the *stereographic projections.*

Note. You may see the stereographic projection in graph theory when considering planar graphs. See my online notes for graduate-level Graph Theory 2 (MATH 5450) on Section 10.1. Plane and Planar Graphs; see Note 10.1.D.

Note. In Exercise I.6.4 it is shown that (1) a circle on S not containing N is mapped to a circle in $\mathbb{C}$, and (2) a circle on S containing N is mapped to a line in $\mathbb{C}_\infty$. In Section III.3. Analytic Functions as Mapping, Möbius Transformations we will discuss lines as if they are circles passing though infinity.

Definition. For $z, z' \in \mathbb{C}_\infty$, define the *distance* from z to z' as the distance between the corresponding points Z and Z' on S (as measured in $\mathbb{R}^3$). Then

$$\begin{aligned}
d(z,z')^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 \\
&= x_1^2 - 2x_1x_1' + (x_1')^2 + x_2^2 - 2x_2x_2' + (x_2')^2 + x_3^2 - 2x_xx_3' + (x_3')^2 \\
&= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3') \\
&= 2 - 2\left\{\left(\frac{z+\overline{z}}{|z|^2+1}\right)\left(\frac{z'+\overline{z}'}{|z'|^2+1}\right) + \left(\frac{-i(z-\overline{z})}{|z|^2+1}\right)\left(\frac{-i(z'-\overline{z}')}{|z'|^2+1}\right)\right. \\
&\qquad \left. + \left(\frac{|z|^2}{|z|^2+1}\right)\left(\frac{|z'|^2-1}{|z'|^2}\right)\right\} \\
&= \cdots = \frac{4|z-z'|^2}{(1+|z|^2)(1+|z'|^2)}, \text{ and so}
\end{aligned}$$

$$d(z,z') = \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}.$$

With $Z' = N = (0,0,1)$, we have $d(z,\infty) = \dfrac{2}{\sqrt{1+|z|^2}}$.

Note. We consider metric spaces in Chapter II, "Metric Spaces and the Topology of $\mathbb{C}$," and it is to be shown in Exercise II.1.7 that $d : \mathbb{C}_\infty \to [0,2]$ is a metric.

Note. The original motivation for stereographic projection is to create a flat map of the night sky. It was probably known to the Egyptians, and likely known to the Greek mathematician Hipparchus (190 BCE–120 BCE), who gave an early example of a trigonometric table and studied triangles on a sphere (for more on Hipparchus, see my online notes for History of Mathematics [MATH 3040] on Section 6.5. Hipparchus, Menelaus, Ptolemy, and Greek Trigonometry; see Notes 6.5.B and 6.5.C). However, the earliest known surviving work on it is by Ptolemy. Claudius Ptolemy (circa 85 CE–165 CE) was a Greek astronomer and geographer whose geocentric theory, as explained in his *Almagest*, dominated astronomy for 1400 years (until it was replaced by heliocentrism, as given by Nicolaus Copernicus [February 19, 1473–May 24, 1543]).

An imagined likeness of Ptolemy from the MacTutor biography webpage of Ptolemy (accessed 9/9/2023).

Ptolemy's work that presents stereographic projection is his *Planisphaerium*. Unlike our approach given above of projecting through the north pole N, Ptolemy considers projecting through the south pole; this is because his focus is on the northern part of the celestial sphere. Ptolemy is aware that the projection maps circles on the sphere to circles or lines in the plane, though he gives no formal proof of this. No versions in the original Greek survive, and *Planisphaerium* is only known through Latin translations of Arabic versions.