

Chapter II. Metric Spaces and the Topology of \mathbb{C}

Note. In this chapter we study, in a general setting, a “space” (really, just a set) in which we can measure “distance.” This simple idea of measuring distance allows us to do lots of analysis stuff (open sets, closed sets, compact sets, limits, continuity, convergent sequences, completeness, etc.).

II.1. Definitions and Examples of Metric Spaces

Note. We introduce metric spaces and give some examples not in the text which you may encounter in other analysis classes. In addition, we briefly discuss topological spaces.

Definition. A *metric space* is a pair (X, d) where X is a set and d is a function mapping $X \times X$ into \mathbb{R} called a *metric* such that for all $x, y, z \in X$ we have

$$d(x, y) \geq 0$$

$$d(x, y) = d(y, x) \text{ (Symmetry)}$$

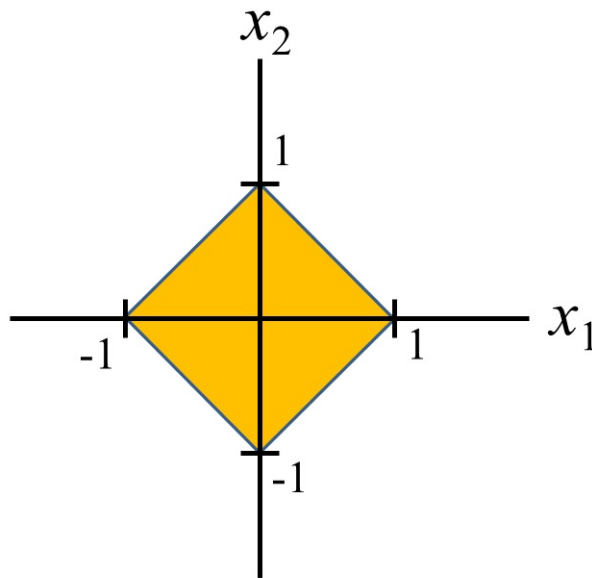
$$d(x, y) = 0 \text{ if and only if } x = y$$

$$d(x, z) \leq d(x, y) + d(y, z) \text{ (Triangle Inequality)}$$

For a given $x \in X$ and $r > 0$, define the *open ball* of center x and radius r as $B(x; r) = \{y \mid d(x, y) < r\}$. Define the *closed ball* of center x and radius r as $\bar{B}(x; r) = \{y \mid d(x, y) \leq r\}$.

Example II.1.1. (\mathbb{R}, d) where $d(x, y) = |x - y|$ is a metric space. Of prime importance for us in metric space (\mathbb{C}, d) where $d(z, w) = |z - w|$.

Example. (\mathbb{R}^2, d) where $d((x_1, x_2), (y_1, y_2)) = |y_1 - x_1| + |y_2 - x_2|$ is a metric space. d is the *taxicab metric*. Notice that the closed unit ball has an unexpected appearance under this metric:



Example. (\mathbb{R}^n, d) where

$$d(\vec{x}, \vec{y}) = d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \left\{ \sum_{k=1}^n |y_k - x_k|^p \right\}^{1/p}$$

is a metric space for $1 \leq p < \infty$. For $p = 2$, this is the usual Euclidean metric in \mathbb{R}^n . (For $0 < p < 1$, the quantity d can still be considered, but it fails the Triangle Inequality.)

Example. (\mathbb{C}^n, d) where

$$\begin{aligned} d(\vec{z}, \vec{w}) &= d((z_1, z_2, \dots, z_n), (w_1, w_2, \dots, w_n)) = \left\{ \sum_{k=1}^n |w_k - z_k|^2 \right\}^{1/2} \\ &= \left\{ \sum_{k=1}^n (w_k - z_k) \overline{(w_k - z_k)} \right\}^{1/2} \end{aligned}$$

is a metric space. In fact, this is an inner product space (an “inner product” is a dot product like encountered in Linear Algebra). That is, \mathbb{C}^n is a vector space with inner product as follows: For $\vec{z} = (z_1, z_2, \dots, z_n)$ and $\vec{w} = (w_1, w_2, \dots, w_n)$, we have $\vec{z} \cdot \vec{w} = \sum_{k=1}^n z_k \overline{w_k}$. Then the *norm* of \vec{z} is

$$\|\vec{z}\| = \sqrt{\vec{z} \cdot \vec{z}} = \left\{ \sum_{k=1}^n z_k \overline{z_k} \right\}^{1/2} = \left\{ \sum_{k=1}^n |z_k|^2 \right\}^{1/2}.$$

The metric in terms of the norm is $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. Alternatively, the norm in terms of the metric is $\|\vec{z}\| = d(\vec{0}, \vec{z})$.

Example. Let

$$\ell^2 = \left\{ (x_1, x_2, x_3, \dots) \mid x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}.$$

Then (ℓ^2, d) is a metric space. In fact, ℓ^2 is an infinite dimensional vector space with (“Schauder”) basis $\{\vec{e}_1, \vec{e}_2, \dots\}$ where $\vec{e}_1 = (1, 0, 0, \dots)$, $\vec{e}_2 = (0, 1, 0, \dots)$, $\vec{e}_3 = (0, 0, 1, 0, \dots), \dots$ in some sense, ℓ^2 is the “most useful” infinite dimensional vector space. The inner product is

$$\vec{x} \cdot \vec{y} = \sum_{k=1}^{\infty} x_k y_k,$$

the norm and inner product are:

$$\|\vec{x}\| = \left\{ \sum_{k=1}^{\infty} |x_k|^2 \right\}^{1/2} \quad \text{and} \quad d(\vec{x}, \vec{y}) = \left\{ \sum_{k=1}^{\infty} |x_k - y_k|^2 \right\}^{1/2}.$$

In ℓ^2 , the idea that a vector has “magnitude and direction” is valid. We will use this space to violate a couple of familiar results. In ℓ^2 we construct a closed and bounded set that is not compact (as opposed to the Heine-Borel Theorem) and an infinite bounded set without a limit point (as opposed to Weierstrass’s Theorem). In fact, we can also produce an ℓ^2 space out of complex numbers.

Example. We can also define ℓ^p spaces for $1 \leq p < \infty$. Let

$$\ell^p = \left\{ (x_1, x_2, \dots) \mid x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}.$$

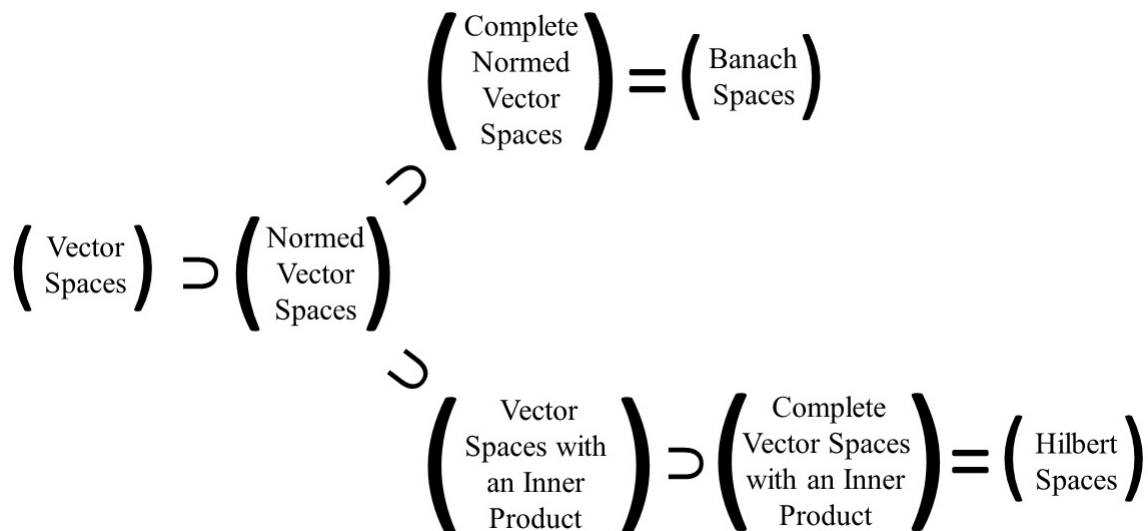
Define

$$d(\vec{x}, \vec{y}) = d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \left\{ \sum_{k=1}^{\infty} |y_k - x_k|^p \right\}^{1/p}.$$

Then (ℓ^p, d) is a metric space. In fact,

$$\|\vec{x}\| = \left\{ \sum_{k=1}^{\infty} |x_k|^p \right\}^{1/p}$$

is a norm. Also, ℓ^p is *complete* in the sense that Cauchy sequences converge (or in the informal sense that there are “no holes” in the space). So the ℓ^p spaces are complete normed vector spaces. Such spaces are called Banach spaces. When $p = 2$, as above, we have that ℓ^2 is a complete inner product space. A complete inner product space is called a Hilbert space. In fact, we have relationships between familiar vector spaces and these new objects as follows:



There are examples of vector spaces which do not have a norm (this would be examples of “nonmetrizable vector spaces”), though they are by our current standards exotic. One might wonder if such vector spaces have bases. They do! Every vector space has a basis in the sense of the term “basis” used in Linear Algebra (using finite linear combinations; such a basis is called a *Hamel basis*), but the proof of this requires Zorn’s Lemma, a result equivalent to the dreaded Axiom of Choice! For a proof, see my supplemental notes for Fundamentals of Functional Analysis (MATH 5740) on [Supplement. Groups, Fields, and Vector Spaces](#) (see Theorem 5.1.4.).

Note. The following is the standard definition of an open set. Remember that if we replace $B(x; \varepsilon)$ with “an open interval $(a - \varepsilon, a + \varepsilon)$ ” then the definition here should reduce to the definitions in senior level analysis (and even to the definitions in Calculus).

Definition II.1.8. A set $G \subset X$ (where (X, d) is a metric space) is *open* if for all $x \in G$ there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset G$.

Note. The book denotes the empty set as \square . In the notes, we still denote it as \emptyset .

Theorem II.1.9. Let (X, d) be a metric space. Then

- (a) X and \emptyset are open,
- (b) if all G_1, G_2, \dots, G_n are open then $\bigcap_{k=1}^n G_k$ is open, and
- (c) if $\{G_j \mid j \in J\}$ is any collection of open sets then $\bigcup_{j \in J} G_j$ is open.

Definition II.1.10. A set $F \subset X$ is *closed* if $X \setminus F$ is open.

Note. DeMorgan's Law allows us to extend Theorem 1.9 to closed sets as follows (see, for example, my online notes for Analysis 1 (MATH 4217/5217) on [Section 3.1. Topology of the Real Numbers](#); notice Theorem 3-3):

Theorem II.1.11. Let (X, d) be a metric space. Then

- (a) X and \emptyset are closed,
- (b) if all F_1, F_2, \dots, F_n are closed then $\bigcup_{k=1}^n F_k$ is closed, and
- (c) if $\{F_j \mid j \in J\}$ is any collection of closed sets then $\bigcap_{j \in J} F_j$ is closed.

Note. We now briefly consider topological spaces. For a detailed exploration, see my online notes for [Introduction to Topology \(MATH 4357/5357\)](#). You can see that the definition is inspired by Theorem I.1.9.

Definition. A *topological space* is a pair (X, \mathcal{T}) where X is a set (of “points”) and \mathcal{T} is a collection of subsets of X such that:

- (a) X and \emptyset are both in \mathcal{T} ,
- (b) if $\{O_1, O_2, \dots, O_n\} \subset \mathcal{T}$ then $\bigcap_{k=1}^n O_k \in \mathcal{T}$, and
- (c) if $\{O_j \mid j \in J\} \subset \mathcal{T}$ then $\bigcup_{j \in J} O_j \in \mathcal{T}$.

An element of \mathcal{T} is said to be *open*. If $A \subset X$ and $X \setminus A \in \mathcal{T}$ then A is *closed*.

Note. In a topological space we may not have an idea of distance (no metric), but we can still talk about limits, continuity, convergence of a sequence, connectedness, etc.

Example. Consider $X = \{a, b, c\}$. A topology on X is $\mathcal{T}_1 = \{X, \emptyset\}$. This is the trivial topology. By the way, the sequence a, a, a, \dots converges to both a and b (and c) in this topological space. A second topology is $\mathcal{T}_2 = \{X, \{a\}, \{b, c\}, \emptyset\}$. The power set $\mathcal{T} = \mathcal{P}(X)$ always forms a topology on X , called the *discrete topology*.

Note. We now return to Conway, but most of the remaining results of this section hold in topological spaces as well.

Definition II.1.12. Let $A \subset X$ in a metric space. The *interior* of A , denoted $\text{int}(A)$, is the union

$$\cup\{G \mid G \text{ is open and } G \subset A\}.$$

The *closure* of A , denoted $A^- = \bar{A} = \text{cl}(A)$ is the set

$$\cap\{F \mid F \text{ is closed and } F \supset A\}.$$

The *boundary* of set A , denoted $\partial(A)$, is the set $A^- \cap (X \setminus A)^-$.

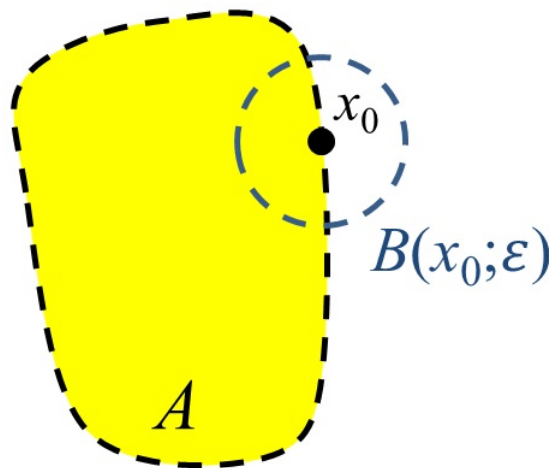
Example. Let $A = \{z \mid |z| \leq 1\} \cup \{z \mid z \in (1, 2)\} \cup \{3\}$. Find the interior of A , the closure of A , and the boundary of A .

Theorem II.1.13. Let $A, B \subset X$. Then

- (a) A is open if and only if $A = \text{int}(A)$,
- (b) A is closed if and only if $A = A^-$,
- (c) $\text{int}(A) = X \setminus (X \setminus A)^-$, $A^- = X \setminus \text{int}(X \setminus A)$, and $\partial(A) = A^- \setminus \text{int}(A)$,
- (d) $(A \cup B)^- = A^- \cup B^-$,
- (e) $x_0 \in \text{int}(A)$ if and only if there exists $\varepsilon > 0$ such that $B(x_0; \varepsilon) \subset A$, and
- (f) $x_0 \in A^-$ if and only if for all $\varepsilon > 0$, $B(x_0; \varepsilon) \cap A \neq \emptyset$.

Note. Parts (a) through (d) only refer to open and closed sets, and so are valid in any topological space. Of course, (e) and (f) make sense in a metric space.

Note. Part (f) of Theorem 1.13 means that $x_0 \in A^-$ if (informally) x_0 is “really close to” set A . So if $x_0 \in A$, then it certainly is close! For $x_0 \notin A$, then *no matter how small ε is*, $B(x_0; \varepsilon)$ still intersects set A :



Definition II.1.14. A subset A of a metric space X is *dense* in X if $A^- = X$.

Example. Familiar examples are: \mathbb{Q} is dense in \mathbb{R} , $\mathbb{R} \setminus \mathbb{Q}$ is dense in \mathbb{R} , and $\{a + ib \mid a, b \in \mathbb{Q}\}$ is dense in \mathbb{C} .

Note. In some settings (namely, Hilbert spaces) it is desirable to have a countable dense subset of a given space. Such a space is called *separable*. See my supplemental notes for Fundamentals of Functional Analysis (MATH 5740) on [Section 5.4](#).

Projections and Hilbert Space Isomorphisms; notice Definition 5.4.1 and Theorem 5.4.8.

Question. The space ℓ^2 (over \mathbb{R} , say) is separable. Give a countable dense subset.

Revised: 2/9/2022