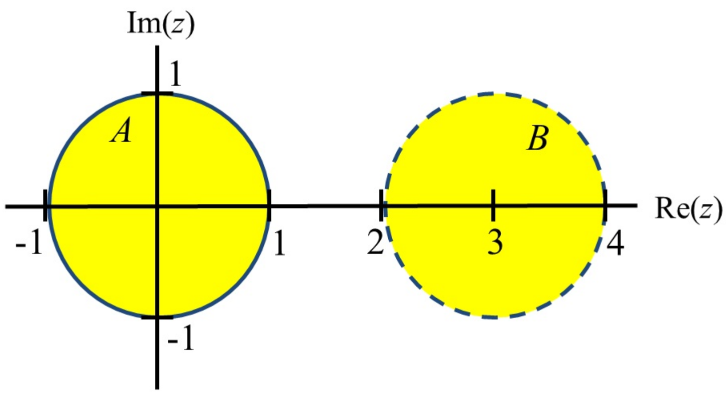
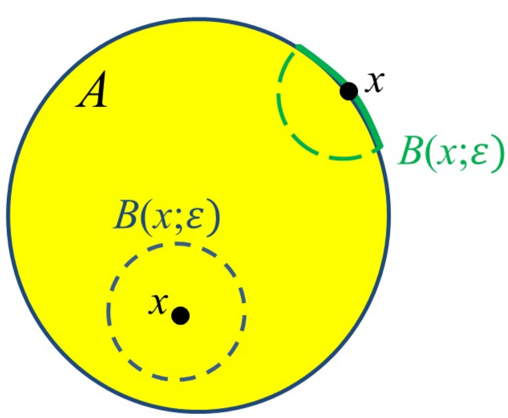
II.2. Connectedness

Note. We now define the idea of connectedness in the metric space setting. Later (Section II.5) we will prove that continuous functions map connected sets to connected sets, and THIS is the real reason such functions are called "continuous"!

Example. Consider the metric space (X, d) where $X = \{z \in \mathbb{C} \mid |z| \leq 1 \text{ or } |z - 3| < 1\}$ and $d(z_1, z_2) = |z_1 - z_2|$:



It is no surprise that A is closed and B is open. However, A is also open! Let $X \in A$. Then there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset A$ (we only need to take $\varepsilon < 1$):



Notice that this means that B is closed. So sets A and B are both open and closed. This is the motivation for the following definition of "connected."

Definition II.2.1. A metric space (X, d) is *connected* if the only subsets of X which are both open and closed are $\varnothing$ and X. If $A \subset X$, then A is a *connected subset* of X if the metric space (A, d) is connected.

Note II.2.A. If (X, d) is not connected, then there are disjoint open sets A and B such that $X = A \cup B$ and $A \neq \varnothing \neq B$. This is similar to the idea of a separation of a set X in $\mathbb{R}$: A *separation* of set $X \subset \mathbb{R}$ is a pair of sets U and V such that:

(1) $U \cap X \neq \varnothing$ and $V \cap X \neq \varnothing$,

(2) U and V are open,

(3) $U \cap V = \varnothing$, and

(4) $X \subset U \cup V$.

A set A is *connected* if there is no separation of the set. In $\mathbb{R}$, the connected sets are intervals and singletons (this is Proposition II.2.2). In the example above, sets A and B form a separation of (X, d). This idea is also studied in Analysis 1 (MATH 4217/5217); see my online notes for Analysis 1 on Section 3.1. Topology of the Real Numbers (notice the definition of "connected set").

Note. The following will be useful when we consider integration.

Definition. For $w, z \in \mathbb{C}$, denote the *line segment* from z to w as

$$[z, w] = \{tw + (1 - t)z \mid 0 \leq t \leq 1\}.$$

A *polygon* from a to b is a set $P = \cup_{k=1}^{n} [z_k, w_k]$ where $z_1 = a$, $w_n = b$, and $w_k = z_{k+1}$ for $1 \leq k \leq n - 1$. We sometimes denote P as $P = [a, z_2, z_3, \ldots, z_{n-1}, b]$.

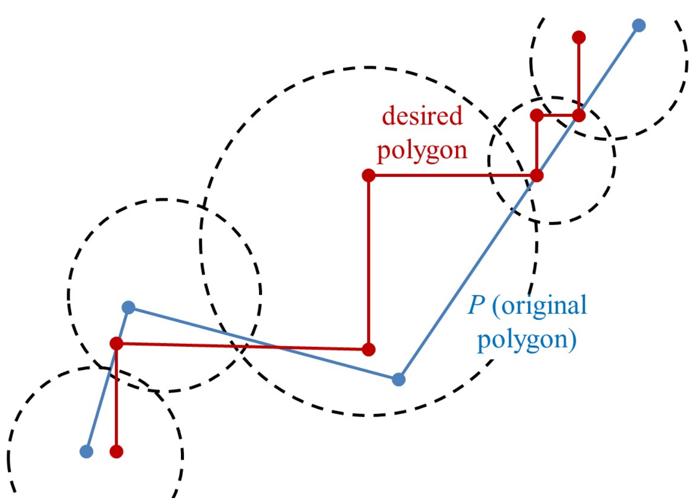
Note. We now state two results which involve the specific metric space $(\mathbb{C}, d)$.

Theorem II.2.3. An open set $G \subset \mathbb{C}$ is connected if and only if for any two points $a, b \in G$ there is a polygon from a to b lying entirely in G.

Note. Notice that the hypothesis of open is necessary in Theorem 2.3. Consider the set $\{z \mid |z| = 1\}$ which is connected but has no (nontrivial) polygonal subsets.

Corollary II.2.4. If $G \subset \mathbb{C}$ is open and connected and a and b are points in G then there is a polygon P in G from a to b which is made up of line segments parallel to either the real or imaginary axis.

Idea of Proof. By Theorem 2.3, there is a polygon P from a to b which lies inside G. Since G is open, for each $x \in P \subset G$, there is $\varepsilon_x > 0$ such that $B(x; \varepsilon_x) \subset G$. Since P is closed and bounded, it is compact (the Heine-Borel Theorem—more of this in Section II.4; this is addressed for $\mathbb{R}$ in Analysis 1 [MATH 4217/5217] in Theorem 3.10/3.11 of Section 3.1. Topology of the Real Numbers), so there are a finite number of open balls covering P: $B(x_1; \varepsilon_1), B(x_2; \varepsilon_2), \ldots, B(x_n; \varepsilon_n)$. "Clearly" we can construct the desired path:



Definition II.2.5. A subset D of a metric space X is a *component* of X if it is a maximal connected subset of X. That is, D is connected and there is no connected subset of X that properly contains D.

Example. Recall that in $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$, a set is open if and only if it is a countable disjoint union of open intervals. This claim is proved in Analysis 1 (MATH 4217/5217); see my online supplemental notes for Analysis 1 on Supplement: A Classification of Open Sets of Real Numbers. The open intervals are the components of the open set.

Lemma II.2.6. Let $x_0 \in X$ and let $\{D_j \mid j \in J\}$ be a collection of connected subsets of X such that $x_0 \in D_j$ for all $j \in J$. Then $D = \cup_{j \in J} D_j$ is connected.

Note. We now show that the components of a set partition the set. Recall that a "partition" of a set is a collection of pairwise disjoint subsets which union to give the larger set. See my online notes of Mathematical Reasoning (MATH 3000) on Section 2.9. Set Decomposition: Partitions and Relations; notice Definition 2.47.

Theorem II.2.7. Let (X, d) be a metric space. Then:

(a) each $x_0 \in X$ is contained in some component of X, and

(b) distinct components of X are disjoint.

Proposition II.2.8.

(a) If $A \subset X$ is connected and $A \subset B \subset A^-$, then B is connected.

(b) if C is a component of X then C is closed.

Note. Now for a result in metric space $(\mathbb{C}, d)$ which describes open sets in $\mathbb{C}$.

Theorem II.2.9. Let G be open in $\mathbb{C}$. Then the components of G are open and there are only a countable number of them.

Revised: 12/6/2023