

II.3. Sequences and Completeness

Note. Probably the most important concept in analysis is that of completeness. Informally, a space, field, linear space, or metric space is complete if it has no holes in it (i.e., if it is a “continuum”). Completeness is necessary if we want to pursue any of the standard topics of analysis such as limits, continuity, and series. The real numbers are a complete ordered field where the Axiom of Completeness states that every nonempty set of real numbers with an upper bound has a least upper bound. However, this definition requires an ordering (a notion of “greater than” and “less than”). That concept does not exist in \mathbb{C} and may not exist in a metric space or vector space. So an alternate approach to completeness must be developed in these settings.

Definition II.3.1. If $\{x_1, x_2, \dots\}$ is a sequence in a metric space (X, d) then $\{x_n\}$ converges to x , denoted $x = \lim x_n$ or $x_n \rightarrow x$, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x, x_n) < \varepsilon$.

Note. We’ll now use sequences to explore several topological properties in metric spaces, including completeness.

Proposition II.3.2. A set $F \subset X$ is closed if and only if for each sequence $\{x_n\}$ in F with $x = \lim x_n$ we have $x \in F$.

Definition II.3.3. If $A \subset X$ then a point $x \in X$ is a *limit point* of A if there is a sequence $\{x_n\}$ of distinct points in A such that $x = \lim x_n$.

Note. The following result links limit points of a set to the closure of a set. It's proof is Exercise II.3.1.

Proposition II.3.4.

- (a) A set is closed if and only if it contains all of its limit points.
- (b) If $A \subset X$ then $A^- = A \cup \{x \mid x \text{ is a limit point of } A\}$.

Definition II.3.5. A sequence $\{x_n\}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. If metric space (X, d) has the property that each Cauchy sequence has a limit in X , then (X, d) is *complete*.

Note. Recall that a sequence of real numbers is Cauchy if and only if it is convergent. By the Triangle Inequality, all convergent sequences are Cauchy. However, the convergence of Cauchy sequences requires the Completeness of \mathbb{R} . In fact, the convergence of Cauchy sequences of real numbers is equivalent to the completeness of \mathbb{R} . This is why we take the above definition of completeness in the metric space setting. We now use this definition to show that \mathbb{C} is complete. The proof is actually quite easy (the hard work is done in the real setting showing that Cauchy sequences converge).

Proposition II.3.6. \mathbb{C} is complete.

Proof. Let $\{x_n + iy_n\}$ be a Cauchy sequence. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m, n > N$ we have

$$d(x_n + iy_n, x_m + iy_m) = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \varepsilon.$$

Then if $m, n \geq N$ we have both $|x_n - x_m| < \varepsilon$ and $|y_n - y_m| < \varepsilon$. So $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences of real numbers. So there are $x, y \in \mathbb{R}$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ (by the completeness of \mathbb{R}). So for all $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that if $n \geq N_1$ then $|x - x_n| < \varepsilon/\sqrt{2}$ and if $n \geq N_2$ then $|y - y_n| < \varepsilon/\sqrt{2}$. So for all $n \geq N_3 = \max\{N_1, N_2\}$ we have:

$$d(x_n + iy_n, x + iy) = \sqrt{(x - x_n)^2 + (y - y_n)^2} < \sqrt{\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2} = \varepsilon.$$

So $\{x_n + iy_n\} \rightarrow x + iy$ and hence \mathbb{C} is complete. ■

Note. The metric space (\mathbb{C}_∞, d) where d is defined in Section I.6 is complete and if $|z_n| \rightarrow \infty$ in \mathbb{C} then $\{z_n\}$ is Cauchy in \mathbb{C}_∞ . This is part of Exercise II.3.4.

Definition. If $A \subset X$, define the *diameter* of A as $\text{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}$.

Note. The above definition is similar to the definition of the diameter of a graph!

Cantor's Theorem. A metric space (X, d) is complete if and only if for any sequence $\{F_n\}$ of nonempty closed sets with $F_1 \supset F_2 \supset F_3 \supset \cdots$ and $\text{diam}(F_n) \rightarrow 0$, then the set $\bigcap_{n=1}^{\infty} F_n$ consists of a single point.

Note. Consider the metric space (X, d) where $X = \mathbb{R} \setminus \{0\}$ and $d(x, y) = |x - y|$. Then this space is not complete. So we can construct a sequence $F_n = [-1/n, 0) \cup (0, 1/n]$ of closed nested sets such that $\text{diam}(F_n) = 2/n \rightarrow 0$, but $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Proposition II.3.8. Let (X, d) be a complete metric space and let $Y \subset X$. Then (Y, d) is a complete metric space if and only if Y is closed in X .

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