

II.4. Compactness

Note. Conway states on page 20 that the concept of compactness is an extension of benefits of finiteness to infinite sets. I often state this idea as: “Compact sets allow us to transition from the infinite to the finite.”

Definition II.4.1. A subset K of a metric space X is *compact* if for every collection \mathcal{G} of open sets in X with the property $K \subset \cup_{G \in \mathcal{G}} G$, there is a finite number of sets G_1, G_2, \dots, G_n in \mathcal{G} such that $K \subset \cup_{k=1}^n G_k$. The collection \mathcal{G} is called an *open cover* of K .

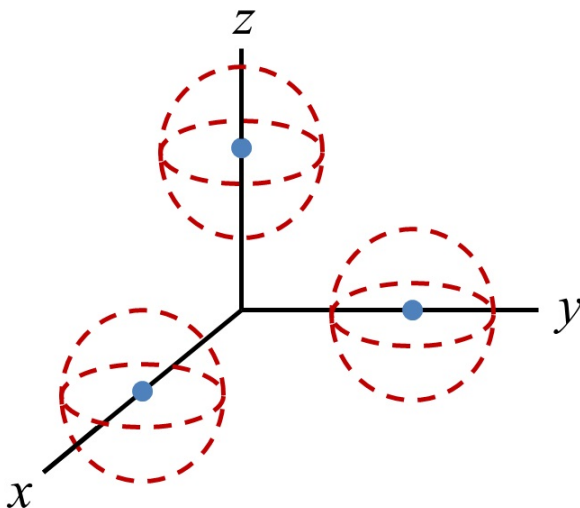
Note. We can paraphrase the definition of compact as: “Every open cover has a finite subcover.” You may recall from senior level analysis that a set of real numbers is compact if and only if it is closed and bounded (the Heine-Borel Theorem). The Heine-Borel Theorem holds in \mathbb{C} (and \mathbb{R}^n and \mathbb{C}^n), but it does not hold in all metric spaces. **BEWARE!** The important property of a compact set is that every open cover has a finite subcover, not something else (like “closed and bounded” which, in general, is not true for compact sets). In fact, let’s violate Heine-Borel in ℓ^2 !

Example. In ℓ^2 , consider the set

$$H = \{(1, 0, 0, \dots), (0, 1, 0, 0, \dots), (0, 0, 1, 0, 0, \dots), \dots\}.$$

Then the distance between any two distinct points in H is $\sqrt{2}$. So all points of H are isolated and hence H is closed (if you like, consider the complement of H).

Since each element of H is distance 1 from the “origin” $(0, 0, 0, \dots)$, then H is bounded. However, H is not compact! Consider the open cover $\mathcal{G} = \{B(x; 1/2) \mid x \in H\}$. Since the distance between any two distinct points of H is $\sqrt{2}$, then $B(x; 1/2) \cap B(y; 1/2) = \emptyset$ for $x \neq y$. So no set in \mathcal{G} can be eliminated from \mathcal{G} and the result still cover H . Therefore there is no finite subcover of \mathcal{G} , and H is not compact. The reason this example works in ℓ^2 (and not in \mathbb{R}^n or \mathbb{C}^n) is because of the infinite number of “directions” (i.e., axes) in ℓ^2 . Consider the first three elements of H and \mathcal{G} :



Imagine this example extended along infinitely many axes and you will get the idea of how the construction works in ℓ^2 .

Note. In fact, in a normed linear space, the closed unit ball is compact if and only if the dimension of the normed linear space is finite. For a proof, see my online notes for Fundamentals of Functional Analysis on [Section 2.8. Finite Dimensional Normed Linear Spaces](#); notice Theorem 2.34.

Note. Set H is also an example of an infinite bounded set with no limit point. Recall that Weierstrass' Theorem says that an infinite bounded set of real numbers (or elements of \mathbb{R}^n or \mathbb{C}^n) has a limit point.

Proposition II.4.3. Let K be a compact subset of X . Then

- (a) K is closed, and
- (b) if F is closed and $F \subset K$ then F is compact.

Note. In fact, we can show that, in a metric space, a compact set is closed AND bounded (by “bounded” we mean that a set A satisfies $A \subset B(x; K)$ for some $x \in X$ and some $K \in \mathbb{R}$). If A is not bounded then the open cover $\{B(x; N) \mid N \in \mathbb{N}\}$ of A (where x is some element of X) has no finite subcover. That is, A is not compact. Hence, if A is compact then it is bounded.

Definition. If \mathcal{F} is a collection of subsets of X such that whenever $\{F_1, F_2, \dots, F_n\} \subset \mathcal{F}$ we have $F_1 \cap F_2 \cap \dots \cap F_n \neq \emptyset$, then \mathcal{F} has the finite intersection property (denoted “f.i.p.”).

Proposition II.4.4. A set $K \subset X$ is compact if and only if every collection \mathcal{F} of closed subsets of K with the finite intersection property satisfies $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

Note. The following two results are useful corollaries which follow from the finite intersection property result and they illustrate the use of Proposition 4.4

Corollary II.4.5. Every compact metric space is complete.

Note. The following may remind you of Weierstrass' Theorem.

Corollary II.4.6. If X is a compact set in a metric space, then every infinite set has a limit point in X .

Note. The following definition makes use of sequences and defines a sequentially compact metric space. We will see that this definition is equivalent to the standard compactness.

Definition II.4.7. A metric space (X, d) is *sequentially compact* if every sequence in X has a convergent subsequence.

Lemma II.4.8. Lebesgue's Covering Lemma.

If (X, d) is sequentially compact and \mathcal{G} is an open cover of X then there is an $\varepsilon > 0$ such that if $x \in X$, there is a set $G \in \mathcal{G}$ with $B(x; \varepsilon) \subset G$.

Note. We now give special conditions equivalent to the compactness of X .

Proposition II.4.9. Let (X, d) be a metric space. The following are equivalent:

- (a) X is compact,
- (b) every infinite subset of X has a limit point,
- (c) X is sequentially compact, and
- (d) X is complete and for all $\varepsilon > 0$ there are a finite number of points $x_1, x_2, \dots, x_n \in X$ such that $X = \cup_{k=1}^n B(x_k; \varepsilon)$. This property is called *total boundedness*.

Note. The following proof of the Heine-Borel Theorem may be a little different from the proof given in your senior level analysis class. This is because we have developed a lot of “heavy equipment” concerning compactness in this section (in particular, the total boundedness of Theorem 4.9). First, we need a technical lemma.

Lemma. Let $F = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ for some real $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ where $a_k < b_k$ for $k = 1, 2, \dots, n$. Then F is totally bounded.

Outline of the Proof. The result follows by proving Exercises 4.2 and 4.3:

Exercise II.4.2. Let $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ be points in \mathbb{R}^n , with $p_k < q_k$ for $k = 1, 2, \dots, n$. Let $R = [p_1, q_1] \times [p_2, q_2] \times \cdots \times [p_n, q_n]$. Then

$$\text{diam}(R) = d(p, q) = \left\{ \sum_{k=1}^n (q_k - p_k)^2 \right\}^{1/2}.$$

Exercise II.4.3. Let $F = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ and let $\varepsilon > 0$. Use Exercise 4.2 to show that there are rectangles R_1, R_2, \dots, R_m such that $F = \cup_{k=1}^m R_k$ and $\text{diam}(R_k) < \varepsilon$ for $k = 1, 2, \dots, m$. Use this to show that F is totally bounded.

Theorem II.4.10. Heine-Borel Theorem.

A subset K of \mathbb{R}^n ($n \geq 1$) is compact if and only if K is closed and bounded.

Note. In Section II.1 of these class notes, we introduced topological spaces. Many of the concepts of this section also hold in the topological space setting. For example, a separation of a set is defined in terms of open sets (see the class notes for Section II.2) and so this definition and hence the definition of connectedness is a topological property. In the previous section (Section II.3) we explored convergence of sequences in a metric space. We can also do this in a topological space, as the following definition shows.

Definition. Let $\{x_n\}$ be a sequence in topological space (X, \mathcal{T}) . Let $x \in X$. If for all open sets $U_x \in \mathcal{T}$ containing x , there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in U_x$, then x is the *limit* of sequence $\{x_n\}$, denoted $\lim x_n = x$.

Note. In a topological space, it may not be the case that the limit of a sequence is unique (it depends on the topology).

Note. In the next section, we will define *continuity* of a function from one topological space to another.

Note. In this section, we approached compactness in a metric space by appealing to Cauchy sequences. One might assume that a similar approach can be taken in a topological space. However, this is not the case as illustrated in an example below. First, we need another topological definition.

Definition. Let (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) be topological spaces. A function $f : X \rightarrow Y$ is a *homeomorphism* if f is continuous, one to one, onto, and has a continuous inverse.

Note. Homeomorphic topological spaces are indistinguishable as topological spaces. A set in \mathcal{T}_1 is open if and only if its image is open in \mathcal{T}_2 . So homeomorphic topological spaces share all “topological” properties.

Example. Let $X = (0, 1)$ and $Y = (1, \infty)$ where \mathcal{T}_1 is the usual topology on X and \mathcal{T}_2 is the usual topology on Y . Then the function $f(x) = 1/x$ is a homeomorphism between (X, \mathcal{T}_1) and (Y, \mathcal{T}_2) . Since each space has the “usual topology,” then these spaces are metrizable (that is, there is a metric which induces the topology; it is the usual metric $d(x_1, x_2) = |x_1 - x_2|$ for each space). However, the sequence $\{x_n\} = \{1/n\}$ is Cauchy in metric space (X, d) but the sequence $\{f(x_n)\} = \{n\}$ is not Cauchy in metric space (Y, d) . So we cannot define a Cauchy sequence in a topological space in a way such that a sequence’s Cauchy-ness is preserved between homeomorphic topological spaces!

Note. John von Neumann published “On Complete Topological Spaces” in 1935 (*Transactions of the American Mathematical Society* **37**(1), 1–20). This is available online on the [AMS website](#) (accessed 2/24/2022). In this paper he addresses the idea of Cauchy sequences in metric spaces and comments: “The need of uniformity in [metric space] M arises from the fact that the elements of a fundamental sequence are postulated to be ‘near to each other,’ and not near to any fixed point. As a general topological space . . . has no property which leads itself to the definition of such a ‘uniformity,’ it is impossible that a reasonable notion of ‘completeness’ could be defined in it.” In this paper, von Neumann discusses total boundedness and compactness in the setting of *topological linear spaces*. His definition of complete is then:

Topological linear space L is *topologically complete* if every closed and totally bounded set $S \subset L$ is compact.

The ‘uniformity’ concern is dealt with by ‘anchoring’ open sets at the origin of the linear space (that is, using the zero vector 0): “However, linear spaces . . . , even if only topological, afford a possibility of ‘uniformization’ for their topology: because of their homogeneity everything can be discussed in the neighborhood of 0 .”

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