II.5. Continuity

Note. We define continuity in the usual ε/δ way. We show the usual properties of continuous functions, including the facts that continuous functions map connected sets to connected sets and compact sets to compact sets. In addition, we define the condition of "Lipschitz" which we explore in a supplement.

Definition II.5.1. Let (X, d) and (Ω, ρ) be metric spaces and let $f : X \to \Omega$ be a function. If $a \in X$ is a limit point of X (so a cannot be an isolated point in X) and $\omega \in \Omega$, then $\lim_{x \to a} f(x) = \omega$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } 0 < d(x, a) < \delta \text{ then } \rho(f(x), \omega) < \varepsilon.$$

Function f is *continuous at point* a if $\lim_{x \to a} f(x) = f(a)$. If f is continuous at each point of X, then f is a *continuous function from* X *to* Ω.

Proposition II.5.2. Let $f : (X, d) \to (\Omega, \rho)$ be a function and $a \in X$, $\alpha = f(a)$. The following are equivalent:

(a) f is continuous at a,

(b) for every $\varepsilon > 0$, $f^{-1}(B(\alpha; \varepsilon))$ contains a ball with center at a, and

(c) $\alpha = \lim_{n \to \infty} f(x_n)$ for all sequences $\{x_n\} \subset X$ with $a = \lim_{n \to \infty} x_n$.

Note. The proof of Proposition 5.2 is assigned as Exercise 5.1. We now shift to continuity on a set as opposed to continuity at a point.

Proposition II.5.3. Let $f : (X, d) \to (\Omega, \rho)$ be a function. The following are equivalent:

(a) f is continuous (on set X),

(b) if Δ is open in Ω then $f^{-1}(\Delta)$ is open in X, and

(c) if Γ is closed in Ω then $f^{-1}(\Gamma)$ is closed in X.

Proposition II.5.4. Let f and g be continuous functions (on X) from X to $\mathbb{C}$ and let $\alpha, \beta \in \mathbb{C}$. Then $\alpha f + \beta g$ and fg are continuous (on X). Also, f/g is continuous provided $g(x) \neq 0$ for all $x \in X$.

Note. The first part of Proposition 5.4 implies that linear combinations of continuous functions are continuous. This foreshadows the idea of making vector spaces out of functions! This is explored in Chapter VII and in great detail in Real Analysis 2 (MATH 5220) in the setting of L^p spaces. The basic idea of "functions as vectors" permeates Fundamentals of Functional Analysis (MATH 5740).

Proposition II.5.5. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$, where $g \circ f(x) = g(f(x))$, is continuous.

Note. Let's step aside briefly and discuss continuity in the setting of topological spaces.

Definition. Let $f : X \to \Omega$ where $(X, \mathcal{T}_1)$ and $(\Omega, \mathcal{T}_2)$ are topological spaces. Then if for $a \in X$ and $\omega \in \Omega$ we have: For each open set $O_\omega \in \mathcal{T}_2$ containing ω there exists an open set $O_1 \in \mathcal{T}_1$ containing a such that

$$\text{for all } x \in O_a, x \neq a, \text{ we have } f(x) \in O_\omega,$$

then we say the *limit* as x approaches a of f is ω, denoted $\lim_{x \to a} f(x) = \omega$.

Note. The above definition is conceptually the same as the ε/δ definition of limit, only with "all open sets" replacing the measures of distances! It may seem surprising that we can discuss limits (which are, informally, descriptions of "getting close") without a concept of distance! It is the *topology* that determines limits of functions. In fact, limits of functions in this setting may not be unique!

Note. We could now define continuity at a point in terms of limits. However, the following definition is more traditional. From it, one can prove that if f is continuous at a then $\lim_{x \to a} f(x) = f(a)$.

Definition. Let $f : X \to \Omega$ where $(X, \mathcal{T}_1)$ and $(\Omega, \mathcal{T}_2)$ are topological spaces. Then f is *continuous* on X if for all open $U \subset \Omega$ we have $f^{-1}(U) \subset X$ is open.

Note. You should not be surprised to see that the equivalence of the limit definition of continuity and the inverse-image-of-open-sets behavior, given in Proposition 5.3, is used to inspire the definition of continuity in the topological space setting. After all, the equivalence of "convergence if and only if Cauchy" and the Axiom of Completeness in $\mathbb{R}$ was the inspiration for our definition of completeness in the metric space setting.

Note. We now return to Conway and metric spaces.

Definition II.5.6. A function $f : (X, d) \to (\Omega, \rho)$ is *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ (depending only on ε) such that $\rho(f(x), f(y)) < \varepsilon$ <u>for all</u> $x, y \in X$ with $d(x, y) < \delta$. Function f is a *Lipschitz function* on X if there is a constant $M > 0$ such that $\rho(f(x), f(y)) \le M\, d(x, y)$ for all $x, y \in X$.

Note. We'll explore Lipschitz functions in more detail in a supplement. For now, notice that if f is Lipschitz on set X then f is uniformly continuous (and hence continuous) on X—just let $\delta = \varepsilon/M$ in the definition of uniform continuity.

Example. The function $f(x) = x^2$ on $X = \mathbb{R}$ is continuous but not uniformly continuous (and hence not Lipschitz). The function $g(x) = \sqrt{x}$ on $X = [0, 1]$ (X is therefore compact) is continuous and uniformly continuous on X, but not Lipschitz on X.

Definition. Let A be a non-empty subset of X and $x \in X$. The *distance* from point x to set A is

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Proposition II.5.7. Let A be a non-empty subset of X. Then:

(a) $d(x, A) = d(x, A^-)$,

(b) $d(x, A) = 0$ if and only if $x \in A^-$, and

(c) $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$.

Note. If we define $f : X \to \mathbb{R}$ as $f(x) = d(x, A)$ for a given set A, then f is Lipschitz since for all $x, y \in X$ we have by Proposition 5.7 (c):

$$|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y).$$

So f is Lipschitz with $M = 1$. Also, f is continuous, a property we will use again in this section.

Note. Some properties of a set *are not* preserved by continuous functions. As you know, a continuous function can map an open set to a non-open set, a closed set to a non-closed set, and a bounded set to an unbounded set. The following result gives two properties of sets that *are* preserved by continuous functions. In fact, the second result is the whole reason continuous functions are called "continuous"!

Theorem II.5.8. Let $f : (X, d) \to (\Omega, \rho)$ be a **C**ontinuous function.

(a) If X is **C**ompact, then $f(X)$ is compact in Ω.

(b) If X is **C**onnected, then $f(X)$ is connected in Ω.

Note. The proof of (b) in Theorem 5.8 only refers to open and closed sets, and so holds as-is in the topological space setting. An easier proof of part (a) which only uses open and closed sets (and Proposition 5.3) is the following.

Alternate Proof of Proposition 5.8 (a).

Let $\mathcal{G}$ be an open cover of $f(X)$. By Proposition 5.3, $f^{-1}(G)$ is open in X for all $G \in \mathcal{G}$. So $\{f^{-1}(G) \mid G \in \mathcal{G}\}$ is an open cover of X. Since X is compact, then there is some finite subcover, say $\{f^{-1}(G_1), f^{-1}(G_2), \ldots, f^{-1}(G_n)\}$. But then $\{G_1, G_2, \ldots, G_n\}$ is a finite subcover of $\mathcal{G}$, and so $f(X)$ is compact. ∎

Note. The following six results show the power of Theorem 5.8 and include some familiar results from Calculus 1 (MATH 1910).

Corollary II.5.9. If $f : X \to \Omega$ is continuous and $K \subset X$ is either compact or connected in X, then $f(K)$ is compact or connected (respectively) in Ω.

Corollary II.5.10. If $f : X \to \mathbb{R}$ is continuous and X is connected, then $f(X)$ is an interval or a singleton.

Proof. This follows from Theorem 5.8 (b), based on the fact that connected sets are intervals or singletons (by Proposition 2.2—Conway considers a singleton an interval: $\{a\} = [a, a]$). ∎

Theorem II.5.11. Intermediate Value Theorem.
If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \le \xi \le f(b)$, then there is a point $x \in [a, b]$ with $f(x) = \xi$.

Proof. By Corollary 5.10 and the fact that $[a, b]$ is connected (by Theorem 2.2), we know that $f([a, b])$ is an interval containing $f(a)$ and $f(b)$. So $f([a, b])$ includes all values "between" $f(a)$ and $f(b)$, including ξ. ∎

Corollary II.5.12. Extreme Value Theorem.
If $f : X \to \mathbb{R}$ is continuous and $K \subset X$ is compact then there are points x_0 and y_0 in K with $f(x_0) = \sup\{f(x) \mid x \in K\}$ and $f(y_0) = \inf\{f(x) \mid x \in K\}$.

Proof. If $\alpha = \sup\{f(x) \mid x \in K\}$ then α is in $f(K)$ since $f(K)$ is compact and hence, by the Heine-Borel Theorem (Theorem 4.10), closed and bounded in $\mathbb{R}$ and so includes its limit points by Proposition 3.4 (a). Similarly, $\beta = \inf\{f(x) \mid x \in K\}$ is in $f(K)$. ∎

Corollary II.5.13. If $K \subset X$ is compact and $f : X \to \mathbb{C}$ is continuous, then there are points x_0 and y_0 in K with $|f(x_0)| = \sup\{|f(x)| \mid x \in K\}$ and $|f(y_0)| = \inf\{|f(x)| \mid x \in K\}$.

Proof. Use Corollary 5.12 with $g : X \to \mathbb{R}$ defined as $g(x) = |f(x)|$. ∎

Corollary II.5.14. If K is a non-empty compact subset of X and $x \in X$, then there is a point $y \in K$ with $d(x, y) = d(x, K)$.

Proof. Define $f : X \to \mathbb{R}$ by $f(y) = d(x, y)$. By the comment after Proposition 5.7, with $A = \{a\}$, we see that $f(y) = d(x, y) = d(y, \{x\})$ is a Lipschitz function on X and hence is continuous. So by Corollary 5.12, f assumes a minimum value on K. That is, there is a point $y \in K$ with $f(y) \leq f(z)$ for all $z \in K$. Since $d(x, K)$ is defined as an infimum, $d(x, K) = d(x, y)$. ∎

Note. The following should be familiar to you in the setting of real domains and codomains for your senior-level Analysis 1 (MATH 4217/5217) class.

Theorem II.5.15. Suppose $f : X \to \Omega$ is continuous and X is compact. Then f is uniformly continuous on X.

Definition II.5.16. For non-empty $A, B \subset X$, define the *distance* from set A to set B as

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

Note. We can have $d(A, B) = 0$ but $A \cap B = \varnothing$. Consider $X = \mathbb{R}^2$, $A = \{(x, y) \mid y = 1/x\}$ and $B = \{(x, y) \mid y = 0\}$. Then $d(A, B) = 0$ but $A \cap B = \varnothing$.

Theorem II.5.17. Let A and B be non-empty disjoint sets in X. If B is closed and A is compact, then $d(A, B) > 0$.

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