II.6. Uniform Convergence

Note. You have probably encountered uniform convergence in your study of Riemann integrals. Recall that if the sequence of Riemann integrable functions $\{f_n\}$ converges uniformly to f on $[a, b]$, then f is Riemann integrable and

$$\lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \int_a^b f(x)\,dx.$$

We now extend the idea of uniform convergence to metric spaces.

Definition. Let X be a set and (Ω, ρ) be a metric space and suppose $f, f_1, f_2, f_3, \ldots$ are functions from X to Ω. The sequence $\{f_n\}$ *converges uniformly* to f, denoted $f = \mathrm{u} - \lim(f_n)$, if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\rho(f(x), f_n(x)) < \varepsilon$ for all $x \in X$. Hence

$$\sup\{\rho(f(x), f_n(x)) \mid x \in X\} \leq \varepsilon$$

for all $n \geq N$.

Theorem II.6.1. Suppose $f_n : (X, d) \to (\Omega, \rho)$ is continuous for each $n \in \mathbb{N}$ and suppose $f = \mathrm{u} - \lim(f_n)$. Then f is continuous.

Note. We will use the following in the next section when addressing the radius of convergence of a complex power series.

Theorem II.6.2. Weierstrass M-Test.

Let $u_n : X \to \mathbb{C}$ be a function such that $|u_n(x)| \leq M_n$ for all $x \in X$ and suppose the constants satisfy $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum_{n=1}^{\infty} u_n$ is uniformly convergent.

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