Chapter III. Elementary Properties and Examples of Analytic Functions

III.1. Power Series

Note. We will see that classical complex analysis is about the study of functions with power series representations (Chapter III) and path integrals of such functions (Chapter IV). In this section, we define the convergence and absolute convergence of a series of complex numbers, and describe the types of sets on which a complex power series converges. This leads to the definition of "radius of convergence" and the Ratio Test. We'll use this information to define the complex exponential function and address products of power series.

Definition. The series $\sum_{n=0}^{\infty} a_n$ converges to z if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \geq N$, $\left| \sum_{n=0}^{m} a_n - z \right| < \varepsilon$. That is, the sequence of partial sums $s_m = \sum_{n=0}^{m} a_n$ converges to z. The series $\sum a_n$ *converges absolutely* if the series of real numbers $\sum |a_n|$ converges.

Note. We see in Analysis 2 (MATH 4227/5227) that an absolutely convergent series of real numbers is convergent. See my online notes for Analysis 2 on Section 7.2. Operations Involving Series (notice Theorem 7-10). In fact, you likely even see this in Calculus 2 (MATH 1920) when covering series; see my online Calculus 2 notes on Section 10.6. Alternating Series, Absolute and Conditional Convergence, and notice Theorem 10. The same result holds for complex series, as the following demonstrates.

Proposition III.1.1. If $\sum a_n$ converges absolutely, then the series converges.

Note. As you see in Calculus 2 (MATH 1920), there are convergent sequences of real numbers that are not absolutely convergent. For example, the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as can be shown with the Integral Test; see my Calculus 2 notes on Section 10.3. The Integral Test). However, the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (as can be shown by the Alternating Series Test; see my Calculus 2 notes on Section 10.6. Alternating Series, Absolute and Conditional Convergence; notice Theorem 14). In fact, by considering the power series for $\ln(1-x)$ it can be shown that the alternating harmonic series converges to $\ln 2$.

Note. You should encounter the next definition in Analysis 1 (MATH 4217/5217). See my online notes for Analysis 1 on Section 2.3. Bolzano-Weierstrass Theorem.

Definition. For a sequence $\{a_n\} \subseteq \mathbb{R}$, define $\underline{\lim}\, a_n = \lim_{n\to\infty} (\inf\{a_n, a_{n+1}, \ldots\})$ and $\overline{\lim}\, a_n = \lim_{n\to\infty} (\sup\{a_n, a_{n+1}, \ldots\})$.

Note. The values of $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ could be $+\infty$ or $-\infty$ since these are defined in terms of suprema and infima. In Analysis 1, we see that $\overline{\lim}\, a_n$ is the greatest subsequential limit point for sequence $\{a_n\}$ (that is, there is a subsequence with $\overline{\lim}\, a_n$ as it's limit, and no convergent subsequence has a greater limit); see Exercise 2.3.16 in my online notes for that class on Section 2.3. Bolzano-Weierstrass Theorem. Similarly, $\underline{\lim}\, a_n$ is the least subsequential limit point for sequence.

Note. The values $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ always exist and $\lim\, a_n$ exists if and only if $\underline{\lim}\, a_n = \overline{\lim}\, a_n$. This is Corollary 2-17 in Section 2.3. Bolzano-Weierstrass Theorem of Analysis 1 (MATH 4217/5217).

Example III.1.A. Consider $\{a_n\} = \{\sin n\}$ (n in radians). Then $\underline{\lim}\, a_n = -1$ and $\overline{\lim}\, a_n = 1$. The proof of this claim is not trivial. A recent graduate of the ETSU Mathematical Sciences Master's Program, Abderrahim Elallam, presented a proof of this in his thesis *Constructions & Optimization in Classical Real Analysis Theorems* (May 2021). In his Section 2.3, "Constructions in the Bolzano-Weierstrass Theorem," be proved:

Proposition 2.1. For every $\alpha \in [-1, 1]$ there is a subsequence $\{x_{n_k}\}$ of $\{x_n = n\}$ such that $\lim_{k\to\infty} \sin(x_{n_k}) = \alpha$.

He lists as a reference for this result G. H. Hardy and E. M. Wright's *An Introduction to the Theory of Numbers* (Oxford University Press, 1981). You can see Mr. Elallam's thesis online through the Digital Commons @ East Tennessee State University (accessed 9/18/2023).

Note. Recall that x is a subsequential limit of sequence $\{a_n\}$ of real numbers if and only if for all $\varepsilon > 0$ the interval $(x-\varepsilon, x+\varepsilon)$ contains infinitely many terms of $\{a_n\}$. This is Theorem 2-11 in my online Analysis 1 (MATH 4217/5217) on Section 2.2. Subsequences. This allows us to create sequences with lots of subsequential limits. Consider the following back-and-forth dialogue:

Question 1. Can you find a sequence with every natural number as a subsequential limit?

Answer. YES! Consider $\{1; 1, 2; 1, 2, 3; 1, 2, 3, 4; \ldots\}$.

Question 2. Can you find a sequence with every rational number as a subsequential limit?

Answer. YES! Let $\{q_n\}$ be an enumeration of the rationals and consider $\{q_1; q_1, q_2;\ q_1, q_2, q_3; q_1, q_2, q_3, q_4; \ldots\}$.

Question 3. Can you find a sequence with every real number as a subsequential limit?

Answer. YES! Take the sequence $\{q_n\}$ as above and use an ε argument. (Notice that for this sequence, $\underline{\lim}\, q_n = -\infty$ and $\overline{\lim}\, q_n = \infty$.)

Note. We now shift our attention to power series in the complex realm. This idea will form the backbone of much of what we do in the the rest of Chapter III.

Definition. A *power series* about $a \in \mathbb{C}$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n.$$

A *geometric series* is of the form $\displaystyle\sum_{n=0}^{\infty} z^n$.

Note. Since $(1 - z^{n+1}) = (1 - z)(1 + z + z^2 + \cdots + z^n)$ then $1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1 - z)$ for $z \neq 1$. If $|z| < 1$ then $\lim_{n\to\infty} z^n = 0$ and so

$$\sum_{n=0}^{\infty} z^n = \lim_{N\to\infty} \left(\sum_{n=0}^{N} z^n \right) = \lim_{N\to\infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z}.$$

If $|z| > 1$ then $\lim_{n\to\infty} |z|^n = \infty$ and $\sum_{n=0}^{\infty} z^n$ diverges. Notice that if $z = 1$ then $\sum_{n=0}^{\infty} z^n = 1 + 1 + 1 + \cdots = \infty$, so the series diverges to infinity. Notice that if $z = -1$ then $\sum_{n=0}^{\infty} z^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$, so the series diverges because the sequence of partial sums oscillates between 0 and 1: $s_0 = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0, \ldots$. So we see that there is ambiguity in the behavior of this series when $|z| = 1$. Recall that a series of real numbers can diverge in two fundamental ways: it can (1) diverge to infinity (or negative infinity), or (2) it can diverge because it bounces around and doesn't "get close" to any set value. This is the case for $\sum_{n=0}^{\infty} z^n$ when $z = 1$ and $z = -1$, respectively.

Theorem III.1.3. If $\displaystyle\sum_{n=0}^{\infty} a_n(z - a)^n$, define the number R as $\frac{1}{R} = \overline{\lim}\, |a_n|^{1/n}$ (so $0 \leq R \leq \infty$). Then

(a) if $|z - a| < R$, the series converges absolutely,

(b) if $|z - a| > R$, the series diverges, and

(c) if $0 < r < R$ then the series converges uniformly on $|z - a| \leq r$. Moreover, R is the only number having properties (a) and (b). R is called the *radius of* *convergence* of the power series.

Note. The following result gives a Ratio Test for complex power series. Compare it to the Ratio Test from Calculus 2 (MATH 1920); see Section 10.5. The Ratio and Root Tests.

Proposition III.1.4. If $\sum_{n=0}^{\infty} a_n(z-a)^n$ is a given power series with radius of convergence R, then $R = \lim |a_n/a_{n+1}|$, if the limit exists.

Note. We can use power series to define functions of a complex variable, provided we know the radius of convergence. Next, we define the "exponential function" with a power series. However, this obliges us to establish that the function behaves in way expected for an exponential function. We will do so in Lemma III.2.A in Section III.2. Analytic Functions.

Definition. Define the *exponential function* $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Note. By Proposition III.1.4, the radius of convergence of e^z is

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \left(\frac{1/n!}{1/(n+1)!} \right) = \infty.$$

Of course when z is real, our complex exponential function is the same as the real exponential function since we see that the power series are the same in both the real and complex settings.

Note. The next two propositions concern the behavior of absolutely convergent series. In Proposition III.1.5, we consider the product of two series of complex numbers (the proof is to be given in Exercise III.1.1). In Proposition III.1.6 we consider the sum and product of two convergent power series in the complex setting. A partial proof is given in the section of the book with details to be added in Exercise III.1.2.

Proposition III.1.5. Let $\sum a_1$ and $\sum b_n$ be two absolutely convergent series and put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum c_n$ is absolutely convergent with sum $\left(\sum a_n\right)\left(\sum b_n\right)$.

Proposition III.1.6. Let $\sum a_n(z-a)^n$ and $\sum b_n(z-a)^n$ be power series with radii of convergence $\geq r > 0$. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then both power series $\sum (a_n + b_n)(z-a)^n$ and $\sum c_n(z-a)^n$ have radius of convergence $\geq r$ and for $|z-a| < r$:

$$\sum (a_n + b_n)(z-a)^n = \sum a_n(z-a)^n + \sum b_n(z-a)^n \text{ and}$$

$$\sum c_n(z-a)^n = \left(\sum a_n(z-a)^n\right)\left(\sum b_n(z-a)^n\right).$$

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