III.2. Analytic Functions

Note. Recall from Analysis 1 (MATH 4217/5217) that a function of a real variable f is *analytic* at $x = c$ if there is an open interval I where $c \in I$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ for all $x \in I$. See my online Analysis 1 notes on Section 8.3. Taylor Series. In this section we give a different definition of an "analytic function" of a complex variable. Our definition, on the surface of it, seems much less restrictive then that from the real setting, but we will ultimately see that our definition implies a power series representation (in the form of a Taylor series) also. In fact, our definition of "analytic" in this section is equivalent to the requirement that the function has a power series representation. So, when you hear "analytic function," think **power series representation!** We start with the definition of a differentiable function.

Definition. If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$, then f is *differentiable at point* $a \in G$ if the following limit exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Note III.2.A. Our definition of "differentiable" at a point is identical with that given in Calculus 1 (MATH 1910); see my online Calculus 1 notes on Section 3.2. The Derivative as a Function. However, both definitions (the real and complex) are based on limits and the limit of a function of a real variable exists if the *two* one-sided limits exist and are equal (that is, the limit from the positive side equals the limit from the negative side). In the complex setting, there are *infinitely many* different directions along which limits must agree. This might motivate the informal idea:

It is "harder" for the limit of a complex function to exist
than it is for the limit of a real function to exist!

An implication of this informal idea is that it is "harder" for a function of a complex variable to be differentiable than it is for a real function to be differentiable. More formally, we see that differentiability at a point implies continuity at that point in the complex setting just as in the real setting, as the following proposition shows.

Proposition III.2.2. If $f : G \to \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a.

Note. Now we state our definition of an analytic function of a complex variable. As mention above, this definition seems to be in conflict with the definition from the real setting. We will resolve this apparent conflict in Theorem IV.2.8 when we show that our definition implies the existence of a power series representation.

Definition. A function $f : G \to \mathbb{C}$ is *analytic* if f is continuously differentiable on G.

Note. All the usual properties of differentiability and continuity hold, so sums, products, and quotients of analytic functions are analytic (with the obvious caveats on quotients). In addition:

Chain Rule. Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and $(g \circ f)'(z) = g'(f(z))f'(z)$ for all $z \in G$.

Note. The next proposition shows, among other things, that a function with a power series representation is continuously differentiable (i.e., analytic); this is explicitly state in Corollary II.2.9 below. This gets us half way to showing that a function of a complex variable is analytic (i.e., continuously differentiable) if and only if it has a power series representation. The proof in the other direction is more complicated and will be given in Theorem IV.2.8 in Section IV.2. Power Series Representations of Analytic Functions.

Proposition III.2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:

(a) for $k \geq 1$ the series

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}$$

has radius of convergence R,

(b) The function f is infinitely differentiable on $B(a; R)$ and the series of (a) equals $f^{(k)}(z)$ for all $k \geq 1$ and $|z - a| < R$, and

(c) for $n \geq 0$, $a_n = \dfrac{1}{n!} f^{(n)}(a)$.

Note. By Proposition III.2.5(b), f is twice differentiable on $B(a; R)$. Hence, f' is differentiable on $B(a; R)$ and by Proposition III.2.2, f' is continuous on $B(a; R)$. Therefore, f is continuously differentiable on $B(a; R)$. We summarize this in the following corollary.

Corollary III.2.9. If the series $\sum a_n(z-a)^n$ has radius of convergence $R > 0$ then $f(z) = \sum a_n(z-a)^n$ is analytic (i.e., continuously differentiable) in $B(a; R)$.

Note III.2.B. We can use Proposition III.2.5 to show that $\dfrac{d}{dz}[e^z] = e^z$ and that this holds for all $z \in \mathbb{C}$. We can simply differentiate the power series that defines e^z term by term using parts (a) and (c) of the proposition.

Proposition III.2.10. If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is constant.

Note. For the proof of Proposition III.2.10, we need two results given in Appendix A. Notice that each deals with functions of a real variable with $\mathbb{C}$ as the codomain.
Proposition A.3. If a function $f : [a, b] \to \mathbb{C}$ is differentiable and $f'(x) = 0$ for all $x \in [a, b]$, then f is a constant.
Proposition A.4. Let $f : [a, b] \to \mathbb{C}$ be a differentiable function. (a) If $g : [c, d] \to [a, b]$ is differentiable then $f \circ g$ is differentiable and $(f \circ g)'(t) = f'(g(t))g'(t)$. (b) If G is an open subset of $\mathbb{C}$ containing $f([a, b])$ and $h : G \to \mathbb{C}$ is an analytic function then $h \circ f$ is differentiable and $(h \circ f)'(x) = h'(f(x))f'(x)$.

Note. By the differentiation property of e^z given in Note III.2.B above and Proposition III.2.10, we can now establish the familiar algebraic properties of the exponential function.

Lemma III.2.A. Properties of e^z include:

(a) $e^{a+b} = e^a e^b$,

(b) $e^z \neq 0$ for all $z \in \mathbb{C}$,

(c) $\overline{e^z} = e^{\overline{z}}$, and

(d) $|e^z| = e^{\mathrm{Re}(z)}$.

Note. Just as we defined e^z using a power series, we can define other functions using power series as well, provided we address the question of convergence by finding the radius of convergence. Next, we define $\cos z$ and $\sin z$ using series.

Definition. Define

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \text{ and } \sin z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} z^{2n-1}.$$

Note III.2.C. For $\cos z$ and $\sin z$ the radius of convergence is $R = \infty$. We also have:

$$\frac{d}{dz}[\cos z] = -\sin z$$

$$\begin{aligned}
\frac{d}{dz}[\sin z] &= \cos z \\
\cos z &= \frac{e^{iz}+e^{-iz}}{2} \\
\sin z &= \frac{e^{iz}-e^{-iz}}{2i} \\
\cos^2 z + \sin^2 z &= 1 \text{ for all } z \in \mathbb{C}.
\end{aligned}$$

Note. The absolute convergence of a series of real numbers allows us to rearrange the terms of the series in any way without affecting the sum (see my online Calculus 2 [MATH 1920] notes on Section 10.6. Alternating Series, Absolute and Conditional Convergence and notice Theorem 17, "The Rearrangement Theorem for Absolutely Convergent Series"; also see Theorem 7-15 in my online notes for Analysis 2 [MATH 4227/5227] on Section 7.2. Operations Involving Series). This same result holds in the complex setting. This allows us to consider the absolutely convergent series for e^{iz} (computed by replace z with iz in the series for e^z) and rearrange it to consider the even powers of z separately from the odd powers of z. This gives (from Note III.2.C) that $e^{iz} = \cos z + i \sin z$.

Definition. A function f is *periodic* if for some $c \in \mathbb{C}$ ($c \neq 0$) we have $f(z+c) = f(z)$ for all $z \in \mathbb{C}$.

Note. From Lemma III.2.A(a), we see that e^z has period $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(1) = e^z.$$

So e^z is NOT a one to one function and does not have an inverse.

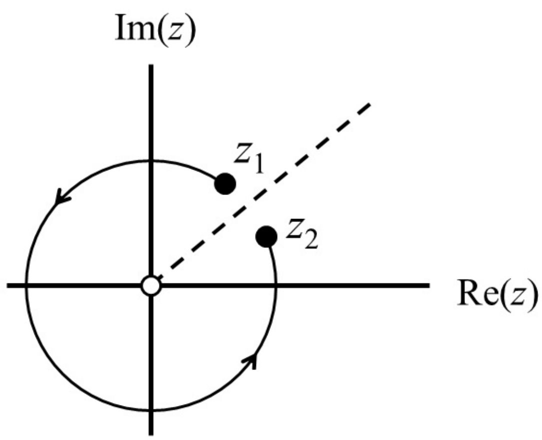
Note. We have that $e^z = w$ if and only if $z = \log|w| + i(\arg(w) + 2\pi k)$ where $k \in \mathbb{Z}$:

$$\begin{aligned} e^{\log|w|+i(\arg(w)+2\pi k)} &= e^{\log|w|}e^{i(\arg(w)+2\pi k)} \\ &= |w|\left(\cos(\arg(w)+2\pi k) + i\sin\left(\arg(w)+2\pi k\right)\right) \\ &= |w|\left(\cos(\arg(w)) + i\sin(\arg(w))\right) = |w|\mathrm{cis}(\arg(w)) = w. \end{aligned}$$

Definition. Let G be an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ be a continuous function such that $z = e^{f(z)}$ on set G. Then f is a *branch of the logarithm* on G.

Note. Since there is no $z \in \mathbb{C}$ such that $e^z = 0$, then $0 \notin G$.

Note III.2.D. We cannot define a branch of the log on $\mathbb{C} \setminus \{0\}$ because of the required continuity property of a branch of the log. We give an informal argument for this, which is to be formalize in Exercise III.2.21.



In this figure, z_1 and z_2 are "close together," but $\arg(z_1)$ and $\arg(z_2)$ must be about 2π apart (to keep continuity on the path) and then we cannot have continuity on a circle around the origin. Therefore, we can only define a branch of the log on $\mathbb{C} \setminus \{0\} \setminus \{$ a path from 0 to $\infty\}$. In much of what follows, we *assume* the existence of a branch of the log and then give its properties or, as in the next result, describe all branches of the log in terms of the given one.

Proposition III.2.19. If $G \subseteq \mathbb{C}$ is open and connected and f is a branch of $\log z$ on G, then the totality of branches of $\log z$ are the functions $\{f(z) + 2k\pi i \mid k \in \mathbb{Z}\}$.

Note III.2.E. One branch of the logarithm is $f(z) = f(re^{i\theta}) = \log r + i\theta$ where $-\pi < \theta < \pi$ and $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$. Notice that with the choice of a branch of the log in this note, we cannot take the logarithm of negative real numbers (though with other branches of the log we could). In a complex variables class, you might use the "principal argument" to define a "principal branch" of the logarithm (see my online notes for Complex Variables [MATH 4337/5337] on Section 3.30. The Logarithm Function). To see that a branch of log is analytic, we need the following.

Proposition III.2.20. Let G and Ω be open subsets of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous where $f(G) \subseteq \Omega$ and $g(f(z)) = z$ for all $z \in G$. If g is differentiable and $g'(z) = 0$, then f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$. If g is analytic, then f is analytic.

Corollary III.2.21. A branch of the log is analytic and its derivative is $1/z$.

Proof. Let $f(z)$ be a branch of log, let $g(z)$ be e^z, and apply Proposition III.2.20.

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Definition. Let $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$ and define the *principal branch of log* of G as given above:

$$f(z) = f(re^{i\theta}) = \log(r) + i\theta \text{ where } -\pi < \theta < \pi.$$

If f is a branch of log on some open connected set H, then $g : H \to \mathbb{C}$ defined as $g(z) = \exp(bf(z))$ is a *branch of* z^b (where $b \in \mathbb{C}$). If we write z^b we will mean the principal branch of z^b (based on the principal branch of $\log z$).

Note. z^b is analytic since e^z and the principal branch of log are analytic. This follows (based on our definition of analytic) from the Chain Rule.

Definition. A *region* is an an open connected set in $\mathbb{C}$.

Note. We now derive a necessary and sufficient condition for function f to be analytic on a region.

Definition. Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$. The *Cauchy-Riemann equations* on u and v are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note. If u and v satisfy the Cauchy-Riemann equations, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ implies } \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \text{ and}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ implies } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Then, if u and v have continuous second partial derivatives (so that second mixed partials are equal), then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Definition. A function $u : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a *harmonic* function.

Theorem III.2.29. Let u and v be real-valued functions defined on a region G and suppose u and v have continuous partial derivatives (so we view u and v as functions of x and y where $z = x + iy$). Then $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if the Cauchy-Riemann equations are satisfied.

Note III.2.F. We see in the proof of Theorem III.2.29 that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ has derivative $f'(z) = u_x(x, y) + iv_x(x, y)$. With the Cauchy-Riemann equations, we can also write $f'(z) = u_x(x, y) - iu_y(x, y) = v_y(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y)$.

Note. There are certainly differences between real and complex functions. For example, $f(x) = |x|^2 = x^2$ is differentiable for all $x \in \mathbb{R}$. However, $f(z) = |z|^2 = x^2 + y^2$ is only differentiable at $z = 0$. (This is Exercise III.2.1.)

Definition. If u and v are harmonic on region G, and if $f = u + iv$ satisfies the Cauchy-Riemann equations, then u and v are *harmonic conjugates*.

Note. The following gives conditions under which u has a harmonic conjugate on G.

Theorem III.2.30. Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is harmonic, then u has a harmonic conjugate on G.

Idea of the Proof. The harmonic conjugate is found as follows:

$$v(x, y) = \int_0^y u_x(x, t)\, dt - \int_0^x u_y(s, 0)\, ds.$$

u and v then satisfy the Cauchy-Riemann equations. $\square$

Revised: 11/7/2023