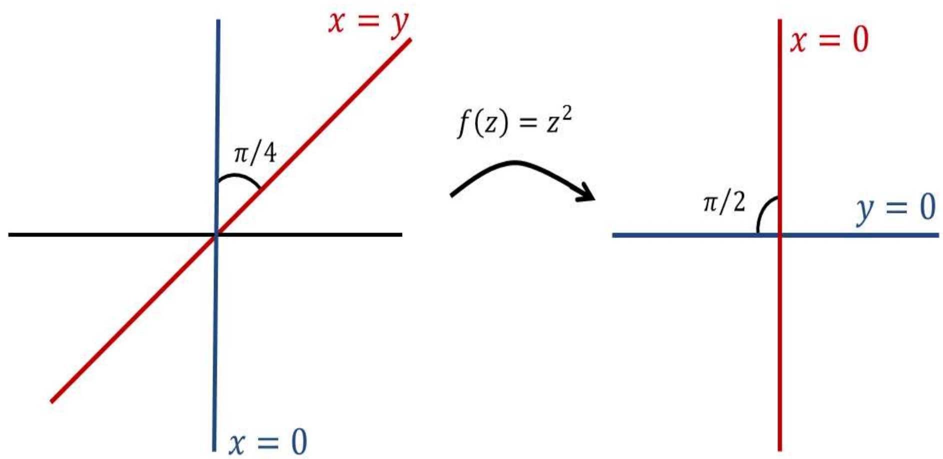
Supplemental Notes on III.3. Analytic Functions as Mappings: Möbius Transformations—with Supplemental Material from Hitchman's *Geometry with an Introduction to Cosmic Topology*

Recall. The/an angle between two smooth paths γ_1 and γ_2 which intersect at $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ is $\arg(\gamma_2'(t_1)) - \arg(\gamma_1'(t_1))$. We have seen some transformations which preserve angles (translations and rotations). We are interested in general in analytic functions which preserve such angles.

Theorem III.3.4. If $f : G \to \mathbb{C}$ is analytic then f preserves angles at each point z_0 of G where $f'(z_0) \neq 0$.

Example. Consider $f(z) = z^2$. Then $f'(z) \neq 0$ unless $z = 0$. If we consider two smooth paths (such as lines) through $z = 0$, we will find that angles between such paths are doubled:

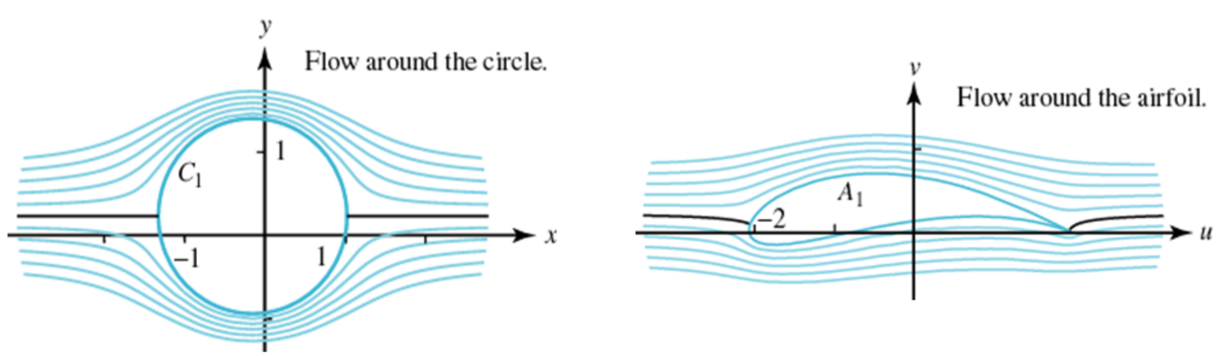


Of course the angle between the x-axis and the x-axis is 0. (or π, depending on how it is parameterized) and doubling this gives 0 (or 2π), so we cannot say that when f' equals 0 that angles are NOT preserved.

Definition. function $f : G \to \mathbb{C}$ such that for paths γ_1 and γ_2 (as above), the angle between γ_1 and γ_2 is the same as the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ is a *conformal map.*

Note. We see that analytic function f is conformal where $f'(z) \neq 0$.

Note. Conformal maps have LOTS of applications. For example, one can study the flow of air over a cylinder or an airfoil:



From: http://mathfaculty.fullerton.edu/mathews/c2003/JoukowskiTransMod.html *(accessed 11/19/2013)*

Definition. If $S(z) = \frac{az+b}{cz+d}$ is a *linear fractional transformation.* If $ad - bc \neq 0$, then $S(z)$ is a *Möbius transformation.*

Note. These transformations are sometimes called *bilinear transformations* and (for some reason) sometimes called *linear transformations.*

Note. We can consider $S(z) = \frac{az+b}{cz+d}$ as a transformation on $\mathbb{C}_\infty$ where $S(\infty) = a/c$ and $S(-d/c) = \infty$.

Note. The composition of two Möbius transformations is a Möbius transformation and for $S(z)$ as above,

$$S^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Therefore the Möbius transformations form a group of transformations on $\mathbb{C}_\infty$, denoted $\mathcal{M}$.

Note. We now supplement the notes with material from *Geometry with an Introduction to Cosmic Topology* by Michael Hitchman, Boston: Jones and Bartlett Publishers (2009). When quoting result's from Hitchman, we present the number of the result and add a prefix of 'H.'

Definition H.4.2.1. *Möbius geometry* is the geometry on space $\mathbb{C}_\infty$ and group $\mathcal{M}$, $(\mathbb{C}, \mathcal{M})$.

Note. For now, we know little about Möbius geometry since we know little about the properties of Möbius transformations.

Notice. In the following, Conway's definition of "inversion" differs from Hitchman's definition (to distinguish between the two, we call Hitchman's inversion an "H-inversion").

Definition. If $S(z) = z + b$ then S is a *translation.* If $S(z) = az$ for $a \in \mathbb{R}$, $a > 0$, then S is a *dilation.* If $S(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$, then S is a *rotation.* If $S(z) = 1/z$, S is an *inversion.*

Notice. Conway's "dilation" is Hitchman's "stretch" (both are misnomers since the translations could be "contractions" or "shrinks").

Theorem III.3.6. If S is a Möbius transformation, then S is a composition of translations, dilations, rotations,and inversions.

Note. An advantage of Hitchman's definition of inversion (or as we call it, "H-inversion") with respect to a cline is the following.

Theorem H.3.4.4. A transformation of $\mathbb{C}_\infty$ is a Möbius transformation if and only if it is the composition of an even number of H-inversions.

Corollary H.3.4.5. Möbius transformations take clines to clines and preserve angles.

Proof. H-inversions map clines to clines by Theorem H.3.2.4. H-inversion preserves angles by Theorem H.3.2.7. The result follows by Theorem H.3.4.4. ∎

Theorem H.3.4.6. (Conway, Page 48.) Any Möbius transformation $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ fixes one, two, or all points of $\mathbb{C}_\infty$.

Note. If $S(a) = \alpha$, $S(b) = \beta$, $S(c) = \gamma$ for distinct a, b, c and if $T(a) = \alpha$, $T(b) = \beta$, $T(c) = \gamma$, then $T^{-1} \circ S$ fixes a, b, c and so $T^{-1} \circ S = I$ and $S = T$. Therefore a Möbius transformation is defined by its action on three distinct points of $\mathbb{C}_\infty$.

Note. The following "cross ratio" stuff is leading into classification of the Möbius transformations relevant to hyperbolic geometry.

Note. For distinct $z_2, z_3, z_4 \in \mathbb{C}_\infty$, define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ as

$$S(z) = \begin{cases} \left(\frac{z-z_3}{z-z_4}\right) \Big/ \left(\frac{z_2-z_3}{z_2-z_4}\right) & \text{if } z_2, z_3, z_4 \in \mathbb{C} \\ \frac{z-z_3}{z-z_4} & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z-z_4} & \text{if } z_3 = \infty \\ \frac{z-z_3}{z_2-z_3} & \text{if } z_4 = \infty. \end{cases}$$

Then $S(z_2) = 1$, $S(z_3) = 0$, and $S(z_4) = \infty$.

Definition III.3.7. For $z_1 \in \mathbb{C}_\infty$, (z_1, z_2, z_3, z_4) is the image of z_1 under the unique transformation which takes z_2 to 1, z_3 to 0, and z_4 to ∞. This is called the *cross ratio* of z_1, z_2, z_3, z_4.

Note. The following is an *invariance* result in that it shows the cross ratio is invariant under Möbius transformations. It becomes Theorem H.3.4.10 of Hitchman when restricted to $\mathbb{C}$.

Theorem III.3.8. If z_2, z_3, z_4 are distinct and T is a Möbius transformation, then

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$$

for all z_1.

Theorem III.3.9. (Theorem H.3.4.7.) Fundamental Theorem of Möbius Transformations.

If z_2, z_3, z_4 are distinct elements of $\mathbb{C}_\infty$ and w_2, w_3, w_4 are distinct elements of $\mathbb{C}_\infty$ then there is one and only one Möbius transformation S such that $Sz_2 = w_2$, $Sz_3 = w_3$, and $Sz_4 = w_4$.

Proof. Let $Tz = (z, z_2, z_3, z_4)$, $Mz = (z, w_2, w_3, w_4)$ and $S = m^{-1}T$. Then $S(z_i) = w_i$ for $i = 2, 3, 4$. If R is another Möbius transformation with $Rz_i = w_i$ for $i = 2, 3, 4$, then $R^{-1}S$ fixes z_2, z_3, z_4 and so $R^{-1}S = I$ and $S = R$. ∎

Note. The composition $S = M^{-1}T$ above reveals how to construct a Möbius transformation between 3 points. Notice that if we want to map one cline to another, we only need to choose 3 points on each and then compute $S = M^{-1}T$.

Note. I quote Conway: "A straight line in the plane will be called a circle." (Page 49) So when Conway says "circle," think of Hitchman's "cline."

Theorem III.3.10. Let z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C}_\infty$. Then (z_1, z_2, z_3, z_4) is real if and only if z_1, z_2, z_3, z_4 lie on a circle (i.e., they lie on a cline).

Note. Conway uses cross ratios to define points symmetric with respect to a cline (ok..., "circle").

Definition. Let Γ be a circle/cline containing distinct points z_2, z_3, z_4 in $\mathbb{C}_\infty$. The points z and z^* in $\mathbb{C}_\infty$ are *symmetric with respect to* Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} = (\overline{z}, \overline{z}_2, \overline{z}_3, \overline{z}_4).$$

Note. See Conway pages 50 and 51 for the computations which justify the fact that Conway and Hitchman are on the same page with the definition of symmetry.

Theorem III.3.19. (Corollary H.3.4.12.) Symmetry Principle.
If a Möbius transformation T takes a circle/cline Γ_1 onto the circle/cline Γ_2 then any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric with respect to Γ_2.

Note. We now put an orientation on a cline and use it to define sides of a cline which are also preserved under Möbius transformations.

Definition III.3.20. If Γ is a cline ("circle") then an *orientation* for Γ is an ordered triple of points (z_1, z_2, z_3) such that each $z_i \in \Gamma$.

Definition. The *right side* of Γ with respect to orientation (z_1, z_2, z_3) is $\{z \mid \text{Im}(z, z_1, z_2, z_3) > 0\}$. The *left side* of Γ with respect to orientation (z_1, z_2, z_3) is $\{z \mid \text{Im}(z, z_1, z_2, z_3) < 0\}$.

Theorem III.3.21. Orientation Principle.
Let Γ_1 and Γ_2 be two circles in $\mathbb{C}_\infty$ and let T be a Möbius transformation such that $T(\Gamma_1) = \Gamma_2$. Let (z_1, z_2, z_3) be an orientation of Γ_1. Then T takes the right side and left side of Γ_1 onto the right side and left side of Γ_2 with respect to the orientation of Γ_2 of (Tz_1, Tz_2, Tz_3).

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