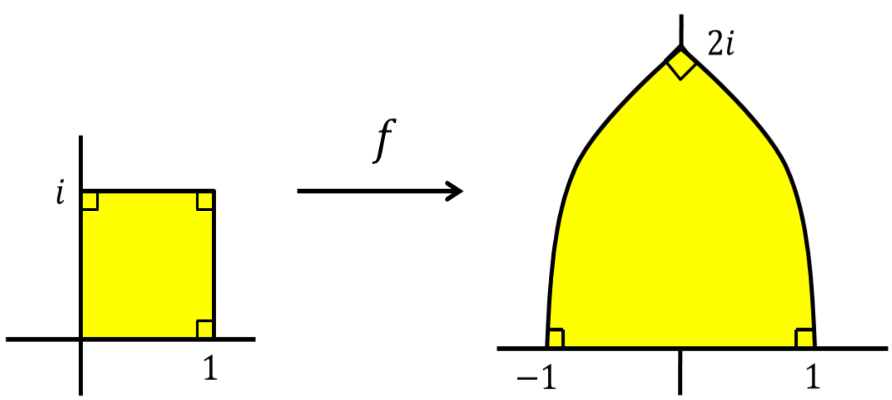
III.3. Analytic Functions as Mapping, Möbius Transformations

Note. To graph $y = f(x)$ where $x, y \in \mathbb{R}$, we can simply plot points (x, y) in $\mathbb{R}^2$ (that is, we can graph $y = f(x)$ is "two dimensional"). However, to graph $w = f(z)$, we need 4 dimensions (2 for the input $z \in \mathbb{C}$ and 2 for the output $z \in \mathbb{C}$). Since we can't "see" this (or even abstractly visualize it), we commonly see how certain sets in $\mathbb{C}$ are mapped under f. In our Complex Variables (MATH 4337/5337) class, the mapping of one region to another is often used in solving applied problems in the setting of conformal mapping.

Example. Consider $f(z) = z^2$ and the set $\{z \mid \text{Re}(z) \in [0, 1], \text{Im}(z) \in [0, 1]\}$. We know that the squaring function doubles arguments and squares moduli. So we have



Except at $(0, 0)$, "angles are preserved."

Note. The problem of mapping one open connected set to another open connected set is a question in a "paramount position" in the theory of analytic functions, according to Conway (page 45). This problem is solved for open simply connected regions in the Riemann Mapping Theorem in Section VII.4.

Definition III.3.1. A *path* in a region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a,b] \to G$ for some interval $[a,b] \subset \mathbb{R}$. If $\gamma'(t)$ exists for each $t \in [a,b]$ and $\gamma' : [a,b] \to \mathbb{C}$ is continuous (that is, $\gamma \in C^1([a,b])$) then γ is a *smooth path*. Also, γ is *piecewise smooth* if there is a partition of $[a,b]$, $a = t_0 < t_1 < \cdots < t_n = b$, such that γ is smooth on each subinterval $[t_{j-1}, t_j]$ for $1 \leq j \leq n$ (smooth on a closed interval means the usual thing on the open interval and differentiability and continuity in terms of one sided limits at the endpoints).

Notes. If $\gamma : [a,b] \to G$ is a smooth path and $t_0 \in (a,b)$ where $\gamma'(t_0) \neq 0$, then we can associate a "direction vector" with the path at point $z_0 = \gamma(t_0)$. The direction vector is the complex number $\gamma'(t_0)$. Notice that an angle $\arg(\gamma'(t_0))$ is associated with this vector, as well as a slope of $\tan(\arg\gamma'(t_0))$.

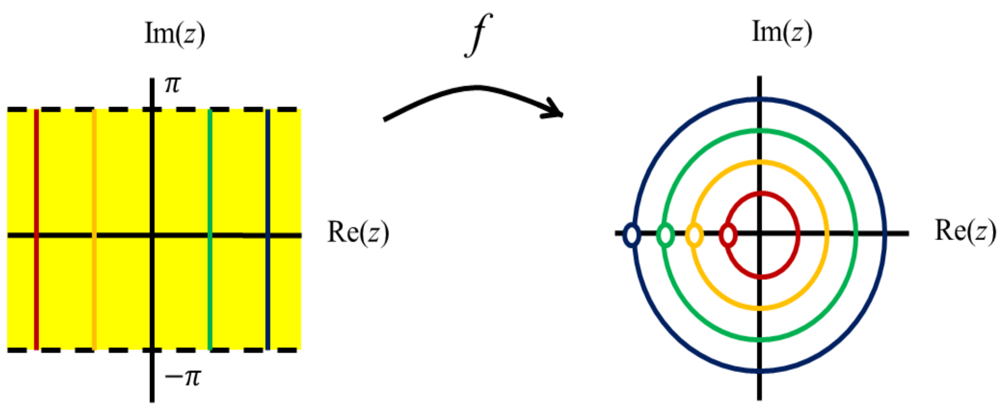
Definition. If γ_1 and γ_2 are smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$, then the (or "an") *angle between the paths* γ_1 and γ_2 at z_0 is $\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1))$.

Theorem III.3.4. If $f : G \to \mathbb{C}$ is analytic then f preserves angles at each point z_0 of G where $f'(z_0) \neq 0$.

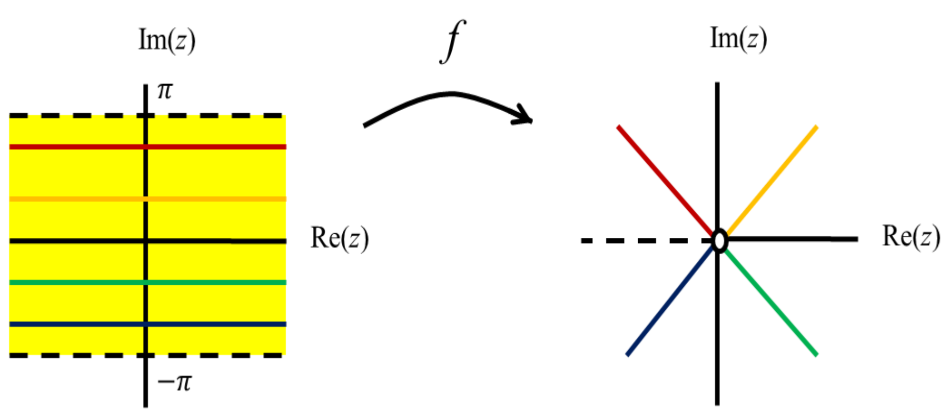
Definition. A function $f : G \to \mathbb{C}$ which preserves angles as described in Theorem 3.4 is a *conformal map*.

Note. So if $f : G \to \mathbb{C}$ is analytic and $f'(z) \neq 0$ for any $z \in G$, then f is conformal.

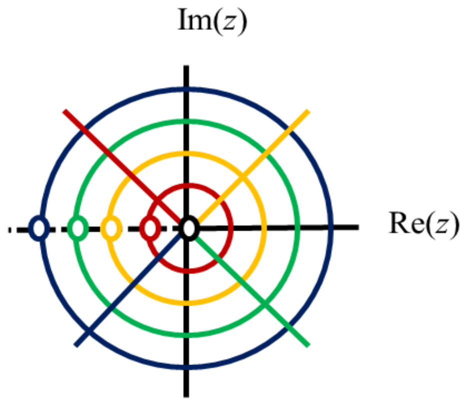
Example. Consider $f(z) = e^z$ defined on $G = \{z \mid -\pi < \text{Im}(z) < \pi\}$. A vertical line segment $x = c$ in G is mapped to a circle in $\mathbb{C}$ of radius e^c:



A horizontal line $y = d$ is mapped to a ray in $\mathbb{C}$ of the form $\{re^{id} \mid 0 < r < \infty\}$:



In fact $f(G) = \mathbb{C} \setminus \{0\}$ and since $f'(z) = e^z \neq 0$, f is conformal on all of G. Combining the above images shows that horizontal and vertical lines which intersect at right angles in G are mapped to rays and circles which intersect at right angles in $\mathbb{C}$:



Of course the principal branch of the logarithm is the inverse of this mapping.

Definition III.3.5. A mapping of the form $S(z) = \dfrac{az+b}{cz+d}$ is a *linear fractional transformation* (or *bilinear transformation*). If $ad - bc \neq 0$ then $S(z)$ is a *Möbius transformation.*

Note. The text calls the Möbius transformations "an amazing class of mappings"! Any Möbius transformation $S(z) = \dfrac{az+b}{cz+d}$ is invertible with $S^{-1}(z) = \dfrac{dz-b}{-cz+a}$. We'll see that Möbius transformations map "circles" to "circles." The collection of Möbius transformations that map the unit disk $D\{z \mid |z| < 1\}$ to itself are fundamental in hyperbolic geometry (see page 85 of Michael Hitchman's *Geometry with an Introduction to Cosmic Topology*, Jones and Bartlett Publishers, 2009).

Note. We can create an onto correspondence between Möbius transformations and the set of 2×2 invertible matrices with complex entries. Consider the association

$$S(z) = \frac{az+b}{cz+d} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can then represent evaluation of the Möbius transformation in terms of matrix multiplication by representing $z = z_1/z_2 \in \mathbb{C}$ as $[z_1, z_2] \in \mathbb{C}^2$. This representation then yields:

$$S(z) = \frac{az+b}{cz+d} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}.$$

This idea is explored in more detail in Exercise III.3.26.

Note. We consider S defined on $\mathbb{C}_\infty$ by defining $S(\infty) = a/c$ (where we interpret a/c as ∞ if $a \neq 0$ and $c = 0$, and $S(\infty) = \infty$ if $c = 0$) and $S(-d/c) = \infty$ (where we interpret $-d/c$ as ∞ when $c = 0$).

Definition. A Möbius transformation of the form $S(z) = z + a$ is a *translation*. If $S(z) = az$ where $a > 0$, then S is a *dilation*. If $S(z) = e^{i\theta}z$ then S is a *rotation*. If $S(z) = 1/z$ then S is an *inversion*.

Proposition III.3.6. If S is a Möbius transformation then S is a composition of translations, dilations, rotations, and inversions.

Proof. If $c = 0$, then $S(z) = (a/d)z + b/d$ and so $S = S_2 \circ S_1$ where $S_1(z) = (a/d)z$ (a rotation and dilation) and $S_2(z) = z + b/d$ (a translation). If $c \neq 0$ then $S = S_4 \circ S_3 \circ S_2 \circ S_1$ where

$S_1(z) = z + d/c$ (a translation),

$S_2(Z) = 1/z$ (an inversion),

$S_3(z) = ((bc - ad)/c^2)z$ (a rotation and dilation), and

$S_4(z) = z + 1/c$ (a translation). ∎

Note. If z is a finite fixed point of Möbius transformation S then

$$S(z) = \frac{az+b}{cz+d} = z \text{ implies } cz^2 + (d-a)z - b = 0.$$

So if $c \neq 0$, then S has two finite fixed points (and ∞ is not fixed by S). If $c = 0$, then S has one finite fixed point and also $S(\infty) = \infty$. If $c = 0$, $d - a = 0$, and $b = 0$ then $S(z) = z$ and all points in $\mathbb{C}_\infty$ are fixed. So, unless $S(z) = z$, Möbius transformation S has at most two fixed points. So the only Möbius transformation with three fixed points is the identity.

Lemma III.3.A. A Möbius transformation is uniquely determined by the action on any three given points in $\mathbb{C}_\infty$.

Proof. Let $a, b, c \in \mathbb{C}_\infty$ be distinct. Suppose S is a Möbius transformation with $\alpha = S(a)$, $\beta = S(b)$, and $\gamma = S(c)$. Suppose T is another Möbius transformation with $\alpha = T(a)$, $\beta = T(b)$, and $\gamma = T(c)$. Then $T^{-1} \circ S$ fixes a, b, and c, so $T^{-1} \circ S$ must be the identity transformation and so $T^{-1} = S^{-1}$ and $S = T$. ∎

Note. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Based on these three points, define Möbius transformation S as

$$S(z) = \begin{cases} \left(\frac{z-z_3}{z-z_4}\right) \Big/ \left(\frac{z_2-z_3}{z_2-z_4}\right) & \text{if } z_2, z_3, z_4 \in \mathbb{C} \\ \frac{z-z_3}{z-z_4} & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z-z_4} & \text{if } z_3 = \infty \\ \frac{z-z_3}{z_2-z_3} & \text{if } z_4 = \infty. \end{cases}$$

Then in each case $S(z_2) = 1$, $S(z_3) = 0$, and $S(z_4) = \infty$. By Lemma III.3.A, this is the only transformation mapping z_2, z_3, z_4 in this way.

Definition III.3.7. If $z_1 \in \mathbb{C}_\infty$ then the *cross ratio* (z_1, z_2, z_3, z_4), where $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct, is the image of z_1 under the unique Möbius transformation S (given above) where $S(z_2) = 1$, $S(z_3) = 0$, and $S(z_4) = \infty$.

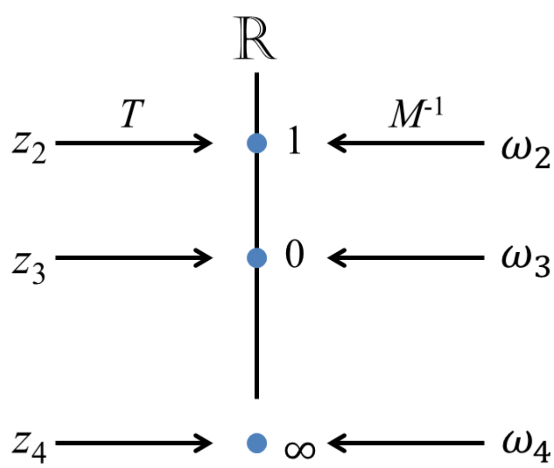
Note. We'll use the cross ratio to show that Möbius transformations preserve certain properties.

Proposition III.3.8. If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct, and T is a Möbius transformation then $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any $z_1 \in \mathbb{C}_\infty$.

Note. The following result shows that any three distinct points in $\mathbb{C}_\infty$ can be mapped to any other three distinct points in $\mathbb{C}_\infty$. In *Geometry with an Introduction to Cosmic Topology*, Hitchman calls this result the "Fundamental Theorem of Möbius Transformations."

Proposition III.3.9. If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct and $\omega_2, \omega_3, \omega_4 \in \mathbb{C}_\infty$ are distinct, then there is one and only one Möbius transformation such that $S(z_2) = \omega_2$, $S(z_3) = \omega_3$, and $S(z_4) = \omega_4$.

Note. The proof of Proposition III.3.9 takes advantage of the "standard set" $\{1, 0, \infty\}$ as follows:



Note. Conway makes the strange statement on page 49 that: "A straight line in the plane will be called a circle." The idea is that a line is a circle of "infinite radius" or, better yet, a circle through ∞. We can then say that three distinct points in $\mathbb{C}$ (or $\mathbb{C}_\infty$) determine a circle. If the points are finite and non-colinear, then this is the old result from Euclidean geometry. If the points are finite and colinear, the "circle" is just the line containing the points. If one of the points is ∞, the "circle" is just the line through the other two points. Hitchman deals with this a bit more cleanly by saying that a *cline* in $\mathbb{C}_\infty$ is either a circle of a line. This comes up in our setting because Möbius transformations map "clines" to "clines" (Theorem III.3.14).

Proposition III.3.10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle/cline.

Theorem III.3.14. A Möbius transformation takes circles/clines onto circles/clines.

Proposition III.3.16. For any given circles/clines Γ and Γ' in $\mathbb{C}_\infty$ there is a Möbius transformation T such that $T(\Gamma) = \Gamma'$. Furthermore we can specify that T take any three points on Γ onto any three points on Γ'. If we do specify Tz_j for $j = 2, 3, 4$ (distinct) then T is unique.

Proof. Let $z_2, z_3, z_4 \in \Gamma$ and $\omega_2, \omega_3, \omega_4 \in \Gamma'$ (distinct). By Proposition III.3.9 there is a unique Möbius transformation T where $Tz_j = \omega_j$ for $j = 2, 3, 4$. By Theorem III.3.14, $T(\Gamma) = \Gamma'$. ∎

Note. We now explore how "interiors" and "exteriors" of circles are mapped under Möbius transformations. First, we need a technical result.

Lemma III.3.B. (Exercise III.3.11.)

Let Γ be a circle/cline in $\mathbb{C}_\infty$. Suppose $z_2, z_3, z_4 \in \Gamma$ are distinct and $\omega_2, \omega_3, \omega_4 \in \Gamma$ are distinct. Then for any $z, z^* \in \mathbb{C}_\infty$ we have $(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$ if and only if $(z^*, \omega_2, \omega_3, \omega_4) = \overline{(z, \omega_2, \omega_3, \omega_4)}$.

Definition III.3.17. Let Γ be a circle/cline in $\mathbb{C}_\infty$ through points z_2, z_3, z_4 (distinct). The points $z, z^* \in \mathbb{C}_\infty$ are *symmetric* with respect Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}. \tag{3.18}$$

Note. Lemma III.3.B shows that the definition of symmetric lines with respect to Γ is independent of the choices of the three points in Γ. Notice that if z is symmetric to itself with respect to Γ then $(z, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$, and so the cross ratio is real and by Proposition III.3.10, $z \in \Gamma$. Conversely, if $z \in \Gamma$ then the cross ratio is real and $z = z^*$, again by Proposition III.3.10. We now put geometric meaning on the idea of symmetry.

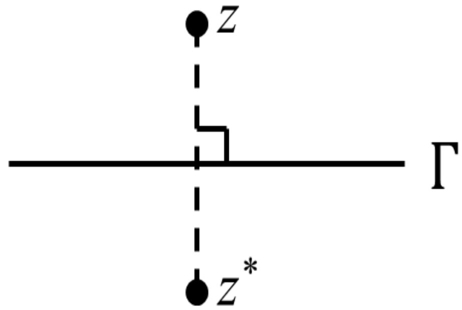
Note. If Γ is a straight line (maybe a "Euclidean line"), then $\infty \in \Gamma$, so with $z_4 = \infty$, equation (3.18) becomes (see page 48):

$$(z^*, z_2, z_3, \infty) = \frac{z^* - z_3}{z_2 - z_3} = \frac{\overline{z} - \overline{z}_3}{\overline{z}_2 - \overline{z}_3} = \overline{(z, z_2, z_3, \infty)}.$$

So we have $\left|\dfrac{z^* - z_3}{z_2 - z_3}\right| = \left|\dfrac{\overline{z} - \overline{z}_3}{\overline{z}_2 - \overline{z}_3}\right|$ or $|z^* - z_3| = |\overline{z} - \overline{z}_3| = |z - z_3|$. Now $z_3 \in \Gamma$ and z_3 is actually arbitrary otherwise, so z and z^* are equidistant from Γ (consider a line through z and z^*, eliminate the possibility that this line is parallel to Γ, then let z_3 be the point of intersection of this line and Γ). Also,

$$\mathrm{Im}\left(\frac{z^* - z_3}{z_2 - z_3}\right) = \mathrm{Im}\left(\frac{\overline{z} - \overline{z}_3}{\overline{z}_2 - \overline{z}_3}\right) = -\mathrm{Im}\left(\frac{z - z_3}{z_2 - z_3}\right). \qquad (*)$$

Now for $w \in \mathbb{C}$ such that $\mathrm{Im}\left(\frac{w - z_3}{z_2 - z_3}\right) = 0$ is a line in $\mathbb{C}$ containing points z_2 and z_3 (see section I.5). So $(*)$ implies that z^* and z lie on opposite sides of Γ (unless they lie on Γ in which case $z = z^*$, as explained above). So, by the equidistant observation above, segment $[z, z^*]$ is a segment perpendicular to Γ and is bisected by Γ. So, if z and z^* are symmetric with respect to straight line Γ, we have



Note. Suppose Γ is a "traditional circle," $\Gamma = \{z \mid |z-a| = R\}$ where $0 < R < \infty$. Let $z_2, z_3, z_4 \in \Gamma$ be distinct. Define $T_1(z) = z - a$, $T_2(z) = \frac{R^2}{z}$, and $T_3(z) = z + a$. Let z and z^* be symmetric with respect to Γ and fixed. Then (3.18) and Proposition III.3.8 imply

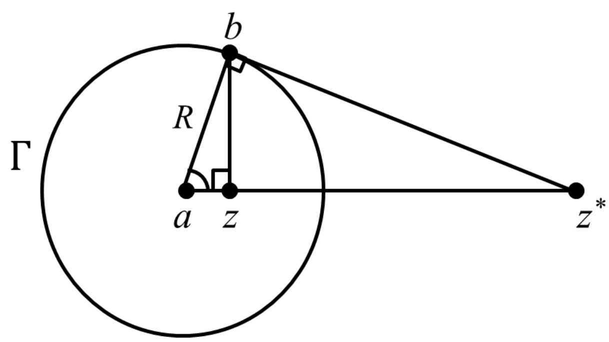
$$\begin{aligned}
(z, z_2, z_3, z_4) &= \overline{(z, z_2, z_3, z_4)} \text{ by (3.18)} \\
&= \overline{(T_1 z, T_1 z_2, T_1 z_3, T_1 z_4)} \text{ by Proposition III.3.8} \\
&= \overline{(z-a, z_2-a, z_3-a, z_4-a)} \text{ by the definition of } T_1 \\
&= (\overline{z}-\overline{a}, \overline{z}_2-\overline{a}, \overline{z}_3-\overline{a}, \overline{z}_4-\overline{a}) \text{ from the definition of cross ratio} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as a Möbius transformation} \\
&= \left(\overline{z}-\overline{a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}, \frac{R^2}{z_4-a}\right) \text{ since } z_j \in \Gamma \text{ and } R^2 = (z_j-a)(\overline{z}_j-\overline{a}) \\
&= \left(T_2\left(\frac{R^2}{\overline{z}-\overline{a}}\right), T_2(z_2-a), T_2(z_3-a), T_2(z_4-a)\right) \text{ by definition of } T_2 \\
&= \left(\frac{R^2}{\overline{z}-\overline{a}}, z_2-a, z_3-a, z_4-a\right) \text{ by Proposition III.3.8}
\end{aligned}$$

$$= \left(T_3\left(\frac{R^2}{\overline{z}-\overline{a}}\right), T_3(z_2-a), T_3(z_3-a), T_3(z_4-a)\right) \text{ by Prop. III.3.8}$$
$$= \left(\frac{R^2}{\overline{z}-\overline{a}}+a, z_2, z_3, z_4\right) \text{ by definition of } T_3.$$

So $z^* = \dfrac{R^2}{\overline{z}-\overline{a}}+a$ (the cross ratio is a Möbius transformation and so is one to one), or $(z^*-a)(\overline{z}-\overline{a}) = R^2$, or $(z^*-a)(\overline{z}-\overline{a})(z-a) = R^2(z-a)$, or $(z^*-a)|z-a|^2 = R^2(z-a)$, or

$$\frac{z^*-a}{z-a} = \frac{R^2}{|z-a|^2} > 0. \qquad (*)$$

So $\mathrm{Im}\left(\dfrac{z^*-a}{z-a}\right) = 0$ and z^* lies on a line through a and with "direction" $z-a$ (see section I.5). That is, $z^* \in \{a+t(z-a) \mid 0 < t < \infty\}$ (t is positive because the right hand side of $(*)$ is positive). Of course, z also lies on this line (when $t=1$). So a, z, z^* are colinear and lie on a ray with a as an endpoint (when $t=0$). Without loss of generality, say z is inside Γ (since $|z-a||z^*-a| = R^2$, we cannot have both z and z^* outside of Γ). Consider:



Then by similar triangles, $\dfrac{|z^*-a|}{R} = \dfrac{R}{|z-a|}$ provided line segment $[z^*, b]$ is tangent to circle Γ. So we see that this gives a geometric meaning to symmetry with respect to a (traditional) circle. It also follows that a and ∞ are symmetric with respect to Γ.

Theorem III.3.19. Symmetry Principle.

If a Möbius transformation takes a circle/cline Γ_1 onto the circle/cline Γ_2 then any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric with respect to Γ_2.

Note. We now put an orientation on a circle/cline which will allow us to discuss left and right sides and show that those are preserved under Möbius transformations.

Definition III.3.20. If Γ is a circle/cline then an *orientation* of Γ is an ordered triple of distinct points (z_1, z_2, z_3) such that each $z_j \in \Gamma$.

Note. Suppose $\Gamma = \mathbb{R}$ and let $z_1, z_2, z_3 \in \mathbb{R}$. Let $Tz = (z, z_1, z_2, z_3) = \frac{az+b}{cz+d}$. Then $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, so by Exercise III.3.8, a, b, c, d can be chosen to be real numbers. Hence

$$Tz = \frac{az+b}{cz+d} = \frac{az+b}{|cz+d|^2}(c\overline{z}+d) = \frac{1}{|cz+d|^2}(ac|z|^2 + bd + bc\overline{z} + adz).$$

So

$$\begin{aligned}\mathrm{Im}(Tz) = \mathrm{Im}(z, z_1, z_2, z_3) &= \frac{1}{|cz+d|^2}\mathrm{Im}(ac|z|^2 + bd + bc\overline{z} + adz)\\ &= \frac{1}{|cz+d|^2}(ad-bc)\mathrm{Im}(z) = \frac{ad-bc}{|cz+d|^2}\mathrm{Im}(z).\end{aligned}$$

Now $\mathrm{Im}(z, z_1, z_2, z_3) = 0$ if and only if $\mathrm{Im}(z) = 0$. That is, $\mathrm{Im}(z, z_1, z_2, z_3) = 0$ is the real axis. So $\mathrm{Im}(z, z_1, z_2, z_3) < 0$ is the lower half plane if $ad - bc > 0$ and the upper half plane if $ad - bc < 0$.

Note. For arbitrary circle/cline Γ, let $z_1, z_2, z_3 \in \Gamma$ be distinct. For any Möbius transformation S we have

$$\begin{aligned}\{z \mid \operatorname{Im}(z, z_1, z_2, z_3) > 0\} &= \{z \mid \operatorname{Im}(Sz, Sz_1, Sz_2, Sz_3) > 0\} \text{ by Prop. III.3.8}\\ &= \{S^{-1}z \mid \operatorname{Im}(z, Sz_1, Sz_2, Sz_3) > 0\} \text{ replacing } z \text{ with } S^{-1}z\\ &= S^{-1}\{z \mid \operatorname{Im}(z, Sz_1, Sz_2, Sz_3) > 0\}.\end{aligned}$$

So if $S : \Gamma \to \mathbb{R}_\infty$, then $\{z \mid \operatorname{Im}(z, z_1, z_2, z_3) > 0\}$ is S^{-1} of either the upper half plane or the lower half plane. This is motivation for the following definition.

Definition. For (z_1, z_2, z_3) an orientation of circle/cline Γ, define the *right side* of Γ with respect to (z_1, z_2, z_3) as $\{z \mid \operatorname{Im}(z, z_1, z_2, z_3) > 0\}$, and the *left side* of Γ as $\{z \mid \operatorname{Im}(z, z_1, z_2, z_3) < 0\}$.

Note. If we travel along Γ from z_1 to z_2 to z_3, then the right side of Γ will lie on our right. We'll further justify this below.

Theorem III.3.21. Orientation Principle.
Let Γ_1 and Γ_2 be two circles/clines in $\mathbb{C}_\infty$ and let T be a Möbius transformation such that $T(\Gamma_1) = \Gamma_2$. Let (z_1, z_2, z_3) be an orientation for Γ_1. <u>Then</u> T takes the right side and the left side of Γ_1 onto the right side and left side of Γ_2 (respectively) with respect to the orientation (Tz_1, Tz_2, Tz_3).

Proof. Let z be on the right side of Γ_1. Then $\mathrm{Im}(z, z_1, z_2, z_3) > 0$. Now by Theorem 3.8, $(z, z_1, z_2, z_3) = (Tz, Tz_1, Tz_2, Tz_3)$, so $\mathrm{Im}(z, z_1, z_2, z_3) = \mathrm{Im}(Tz, Tz_1, Tz_2, Tz_3) > 0$, and Tz is on the right side of $T(\Gamma_1) = \Gamma_2$ with respect to orientation (Tz_1, Tz_2, Tz_3) of Γ_2. Similarly, if z is on the left side of Γ_1, then Tz is on the left side of Γ_2. ∎

Note. Giving $\mathbb{R}_\infty$ the orientation $(1, 0, \infty)$ we have the cross ratio $(z, 1, 0, \infty) = z$. So the right side of $\mathbb{R}_\infty$ with respect to $(1, 0, \infty)$ is the upper half plane $\mathrm{Im}(z) > 0$. If we travel along $\mathbb{R}_\infty$ from 1 to 0 to ∞, then the upper half plane is on our right. This fact, combined with the Orientation Principle, justifies our claim above about left and right sides.

Note. We are interested in general in mapping one set to another with an analytic function. This can be done for many special cases using Möbius transformations. This problem is dealt with in a fairly general setting with the Riemann Mapping Theorem in section VII.4.

Example. We now find a Möbius transformation which maps $G = \{z \mid \mathrm{Re}(z) > 0\}$ onto $\{z \mid |z| < 1\}$. This can be done by mapping the imaginary axis onto the unit circle and using an orientation of the imaginary axis which puts the right half plane on the right side of the imaginary axis, and an orientation of the unit circle which puts the interior on the right. So we use the orientation $(-i, 0, i)$ of the imaginary axis and $(-i, -1, i)$ of the unit circle. Then (from page 48) $(z, -i, 0, i) = \frac{2z}{z-i} = S(z)$ and $(z, -i, -1, i) = \frac{2i}{i-1}\frac{z+1}{z-i} = R(z)$. So S maps the imaginary axis to the real axis and R maps the unit circle to the real axis. So $T = R^{-1} \circ S$ maps the imaginary axis to the unit circle. By the Orientation Principle, we are insured that T is the desired mapping. We find that $T = R^{-1} \circ S = \frac{z-1}{z+1}$.