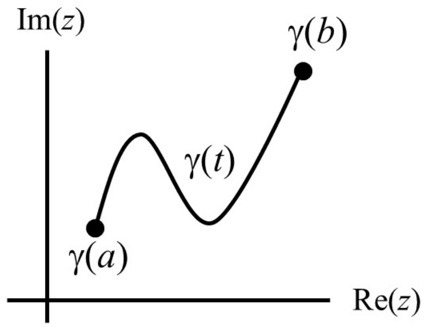
Chapter IV. Complex Integration

IV.1. Riemann-Stieltjes Integrals

Note. In this section, we integrate over paths in the complex plane:



To do so, we need to introduce Riemann-Stieltjes integrals. You may have seen this in Analysis 1 (MATH 4217/5217) for real-valued functions. For example, see my online Analysis 1 notes on Section 6.3. The Riemann-Stieltjes Integral. In the real setting, Riemann-Stieltjes sums are set up by weighing function values by the change in a second function over intervals forming a partition of the interval over which we are integrating. In this way, $\Delta x_k = x_k - x_{k-1}$ of the regular Riemann sum (see my online Calculus 1 notes on Section 5.2. Sigma Notation and Limits of Finite Sums) is replaced with $\Delta g_k = g(x_k) - g(x_{k-1})$. The real Riemann-Stieltjes integral then reduces to the regular Riemann integral with g as the identity function $g(x) = x$. In the complex Riemann-Stieltjes integral, the function determining the path $\gamma(t)$ will play the role of the function g in the real Riemann-Stieltjes integral. We have concern over divergence of complex Riemann-Stieltjes sums, and this leads us to the following definition that requires a certain boundedness.

Definition. A function $\gamma : [a, b] \to \mathbb{C}$, for $[a, b] \subset \mathbb{R}$, is of *bounded variation* if there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < t_2 < \cdots < t_m = b\}$ of $[a, b]$,

$$v(\gamma; P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

The *total variation* of γ, $V(\gamma)$, is defined as

$$V(\gamma) = \sup\{v(\gamma; P) \mid P \text{ is a partition of } [a, b]\}.$$

Note. The next proposition gives some initial properties of variation with respect to a partition, and total variation. The proof is to be given in Exercise IV.1.2.

Proposition IV.1.2. Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation. Then:

(a) if P and Q are partitions of $[a, b]$ and $P \subseteq Q$ then $v(\gamma; P) \leq v(\gamma; Q)$, and

(b) if $\sigma : [a, b] \to \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$ then $\alpha\gamma + \beta\sigma$ is of bounded variation and $V(\alpha\gamma + \beta\sigma) \leq |\alpha|V(\gamma) + |\beta|V(\sigma)$.

Note. In Calculus 2 (MATH 1920), a function is defined as *smooth* if it is differentiable with a continuous derivative. The length of smooth curve $(x(t), y(t))$, where $t \in [a, b]$, is

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Recall that a Riemann integral on $[a, b]$ exists if the integrand is continuous (in fact, the integral can be evaluated using the Fundamental Theorem of Calculus and antiderivatives if the integrand is continuous). This is the desired setting in Calculus 2. If $(x(t), y(t))$ describes the path of a particle in $\mathbb{R}^2$, then the velocity of the particle is the vector $\langle x'(t), y'(t)\rangle$. So (heuristically): length $= \int_a^b$ speed dt. This motivation makes the next proposition unsurprising.

Proposition IV.1.3. If $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth then γ is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Note. The smoothness of γ in Proposition IV.1.3 is necessary, as seen in Exercise IV.1.7 where it is shown that $\gamma(t) = t + it \sin \frac{1}{t}$ for $t \neq 0$ and $\gamma(0) = 0$ is a path on $[0, 1]$ (i.e., it is continuous on $[0, 1]$), but it is not of bounded variation (so it is not continuously differentiable on $[0, 1]$; the right-hand derivative does not exist at 0).

Note. IV.1.A We now use the variation in γ to "weigh" function values and define the Riemann-Stieltjes integral. We will integrate over paths in the complex plan which are parameterized in terms of a real variable from an interval $[a, b]$. Because of this, there is no Riemann integral in the complex setting. The next theorem shows that a continuous function on a path of bounded variation actually has a Riemann-Stieltjes integral (which the theorem formally defines).

Theorem IV.1.4. Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Then there is a complex number I such that for all

$\varepsilon > 0$, there exists $\delta > 0$ such that when $P = \{t_0 < t_1 < \cdots < t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max\{t_k - t_{k-1}\} < \delta$, then

$$\left| I - \sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon$$

for whatever choice of points τ_k, where $\tau_k \in [t_{k-1}, t_k]$. The number I is called the *Riemann-Stieltjes integral* of f with respect to γ over $[a, b]$, denoted

$$I = \int_a^b f\, d\gamma = \int_a^b f(t)\, d\gamma(t).$$

Note. We now state two propositions which have routine proofs. The first concerns the linear behaviors of the Riemann-Stieltjes integral and the second considers additivity of the integral. A proof of Proposition IV.1.7 is to be given in Exercise IV.1.3 and a proof of Proposition IV.1.8 is to be given in Exercise IV.1.4.

Proposition IV.1.7. Let f and g be continuous functions on $[a, b]$ and let γ and σ be functions of bounded variation on $[a, b]$. Then for any complex scalars α and β we have:

(a) $\displaystyle\int_a^b (\alpha f + \beta g)\, d\gamma = \alpha \int_a^b f\, d\gamma + \beta \int_a^b g\, d\gamma$, and

(b) $\displaystyle\int_a^b f\, d(\alpha\gamma + \beta\sigma) = \alpha \int_a^b f\, d\gamma + \beta \int_a^b f\, d\sigma$.

Proposition IV.1.8. Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and let $f : [a, b] \to \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \cdots < t_n = b$ then

$$\int_a^b f\, d\gamma = \sum_{k=1}^{n} \left(\int_{t_{k-1}}^{t_k} f\, d\gamma \right).$$

Note. The following is useful in the computation of integrals. However, we will not in general evaluate integrals in the complex setting by simply using antiderivatives; we'll see that the values of integrals depend on the path γ (and not just its endpoints), as well as the region on which the integrand is analytic. We will see below one way to evaluate complex integrals (see Note IV.1.B).

Theorem IV.1.9. If γ is piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous then

$$\int_a^b f\, d\gamma = \int_a^b f(t)\gamma'(t)\, dt.$$

Definition. If $\gamma : [a, b] \to \mathbb{C}$ is a path (i.e., continuous) then the set $\{\gamma(t) \mid t \in [a, b]\}$ is the *trace* of γ, denoted $\{\gamma\}$. γ is a *rectifiable path* if γ is a function of bounded variation.

Note. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path with $\{\gamma\} \subset E \subset \mathbb{C}$ and $f : E \to \mathbb{C}$ is continuous then $f \circ \gamma$ is continuous and maps $[a, b]$ to $\mathbb{C}$. We are now in position to define an integral of a complex valued function $f(z)$ along a path.

Definition IV.1.12. If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path and f is a function defined and continuous on $\{\gamma\}$ then the (*line*) *integral of f along γ* is

$$\int_a^b f(\gamma(t))\, d\gamma(t),$$

denoted $\displaystyle\int_\gamma f = \int_\gamma f(z)\, dz$.

Note IV.1.B. In order to use the Fundamental Theorem of Calculus for a real-valued function of a real variable, we must (at this stage) partition the integral of Definition IV.1.12 into its real and imaginary parts:

$$\int_a^b f(\gamma(t))\, d\gamma(t) = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b \{\mathrm{Re}(f(\gamma(t))\gamma'(t)) + i\,\mathrm{Im}(f(\gamma(t))\gamma'(t))\}\, dt$$

$$= \int_a^b \mathrm{Re}(f(\gamma(t))\gamma'(t))\, dt + i\int_a^b \mathrm{Im}(f(\gamma(t))\gamma'(t))\, dt,$$

where we have used the linearity condition given in Proposition IV.1.7.

Note. Until we state our "Fundamental Theorem of Calculus" for the the complex setting (in Theorem IV.1.18 below), we use the previous definition when integrating.

Example IV.1.A. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = e^{it}$ and let $f(z) = 1/z$ for $z \neq 0$. Then by Definition IV.1.12:

$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{e^{it}}(ie^{it})\, dt = i\int_0^{2\pi} dt = 2\pi i.$$

Example IV.1.B. Consider again $\gamma : [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = e^{it}$. Let $f(z) = z^n$ for some integer $n \geq 0$. Then by Definition IV.1.12:

$$\int_\gamma z^n\,dz = \int_0^{2\pi} e^{int}(ie^{it})\,dt = i\int_0^{2\pi} e^{i(n+1)t}\,dt = i\int_0^{2\pi} (\cos(n+1)t + i\sin(n+1)t)\,dt$$

$$= i\int_0^{2\pi} \cos(n+1)t\,dt - \int_0^{2\pi} \sin(n+1)t\,dt = 0.$$

Now consider $\sigma(t) = tb + (1-t)a$ where $t \in [0,1]$ (that is, σ represents a straight line segment from a to b). Then $\sigma' = b - a$ and for $f(z) = z^n$, $n \geq 0$, we have by Definition IV.1.12:

$$\begin{aligned}\int_\sigma z^n\,dz &= \int_0^1 (tb + (1-t)a)^n (b-a)\,dt \\ &= (b-a)\int_0^1 (tb + (1-t)a)^n\,dt \text{ by Proposition IV.1.7(a).}\end{aligned}$$

At this stage, we are "stuck" unless we can find the real and imaginary parts of the integrand (see Note IV.1.B). Parameter t is real, but a and b are complex, complicating the calculation. Below we will have a Fundamental Theorem of Calculus for complex functions (see Theorem IV.1.18). This will allow us to evaluate this integral using antiderivatives to find that $\displaystyle\int_\sigma z^n\,dz = \frac{b^{n+1} - a^{b+1}}{n+1}$. See Example IV.1.C below.

Note. If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c,d] \to [a,b]$ is continuous and nondecreasing (here, $[a,b], [c,d] \subset \mathbb{R}$) and $\sigma([c,d]) = [a,b]$ (it follows that $\sigma(c) = a$ and $\sigma(d) = b$) then $\gamma \circ \sigma : [c,d] \to \mathbb{C}$ is a path with the same trace as γ. Then $\gamma \circ \sigma$ is rectifiable as well and if f is continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ then $\int_{\gamma\circ\sigma} f$ is defined as above. We'll use σ in a "u-substitution" setting.

Proposition IV.1.13. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any f continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_\gamma f = \int_{\gamma\circ\sigma} f$.

Definition IV.1.16. Let $\sigma : [c, d] \to \mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C}$ be rectifiable paths. The path σ is *equivalent* to γ if there is a function $\varphi : [c, d] \to [a, b]$ which is continuous, strictly increasing, and with $\varphi(c) = a$, $\varphi(d) = b$, such that $\sigma = \gamma \circ \varphi$. Function φ is a *change of parameter*. A *curve* is an equivalence class of paths.

Note. We usually won't draw the subtle distinction between a path and curve (and the trace of a path).

Definition. For rectifiable $\gamma : [a, b] \to \mathbb{C}$ and for $a \le t \le b$, define $|\gamma|(t)$ as $|\gamma|(t) = V(\gamma; [a, t])$, where

$$V(\gamma; [a, t]) = \sup \left\{ \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \;\middle|\; \{t_0, t_1, \ldots, t_n\} \text{ is a partition of } [a, t] \right\}.$$

Note. The function $|\gamma|(t)$ is like a cumulative variation. Since γ is rectifiable, so is $|\gamma|$.

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be rectifiable and f continuous on $\{\gamma\}$. Define

$$\int_\gamma f\,|dz| = \int_a^b f(\gamma(t))\,d|\gamma|(t).$$

Define $-\gamma(t)$ as $-\gamma(t) = \gamma(-t)$ for $t \in [-b, -a]$.

Note. The next proposition gives some basic properties of the line integral. Notice that there is a real analogue for each claim. The proof is to be given in Exercise IV.1.18.

Proposition IV.1.17. Let γ be a rectifiable curve and suppose that f is a function continuous on $\{\gamma\}$. Then:

(a) $\displaystyle\int_\gamma f = -\int_{-\gamma} f$,

(b) $\displaystyle\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le V(\gamma)\sup\{|f(z)| \mid z \in \{\gamma\}\}$,

(c) if $c \in \mathbb{C}$ then $\displaystyle\int_\gamma f(z)\,dz = \int_{\gamma+c} f(z-c)\,dz$.

Note. Now for our **Fundamental Theorem of Calculus**!

Theorem IV.1.18. Let G be open in $\mathbb{C}$ and let γ be a rectifiable path in G with initial and end points α and β. If $f : G \to \mathbb{C}$ is a continuous function with a *primitive* $F : G \to \mathbb{C}$ (i.e., $F' = f$), then

$$\int_\gamma f = F(\beta) - F(\alpha).$$

Example IV.1.C. As in Example IV.1.B, consider $\sigma(t) = tb + (1-t)a$ where $t \in [0,1]$, and $f(z) = z^n$ where $n \in \mathbb{Z}$, $n \ge 0$. We have by Theorem IV.1.18:

$$\int_\sigma z^n\,dz = \left.\frac{z^{n+1}}{n+1}\right|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Note. Before proving our Fundamental Theorem of Calculus, we need a lemma.

Lemma IV.1.19. If G is an open set in $\mathbb{C}$, $\gamma : [a,b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon$.

Exercise IV.1.10. Define $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$ and find $\int_\gamma z^n \, dz$ for every integer n.

Solution. We use our Fundamental Theorem of Calculus (Theorem IV.1.18), if we can. To use it, we need a primitive (i.e., antiderivative) of z^n that is valid on an open set G containing γ. For $n \neq -1$, a primitive of z^n is $z^{n+1}/(n+1)$. Theorem IV.1.18 then gives

$$\int_\gamma z^n \, dz = \frac{\beta^{n+1}}{n+1} - \frac{\alpha^{n+1}}{n+1} = \frac{(1)^{n+1}}{n+1} - \frac{(1)^{n+1}}{n+1} = 0,$$

since $\alpha = \gamma(0) = e^{i(0)} = 1$ and $\beta = e^{i(2\pi)} = 1$. If $n = -1$, then $z^{-1} = 1/z$ does not have a primitive on G (recall that a branch of the logarithm must have a branch cut, see Note III.2.D in Section III.2. Analytic Functions, and so cannot be continuous on G and cannot be differentiated, so that it is not a primitive on G). So we need a different technique. We can use the method of Note IV.1.B, which is illustrated for this function in Example IV.1.A above, where we saw that $\int_\gamma \frac{1}{z} \, dz = 2\pi i$. $\square$

Definition. A curve $\gamma : [a,b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$.

Note. If we can apply our Fundamental Theorem of Calculus (Theorem IV.1.18) to an integral over a closed curve, then the value of the integral must be 0. Exercise IV.1.10 above illustrates this in the case where $f(z) = z^n$, $n \neq -1$, and $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. We summarize this in the following corollary.

Corollary IV.1.22. Let G, γ, and f satisfy the same hypotheses as in Theorem IV.1.18. If γ is a closed curve, then $\int_\gamma f = 0$.

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