IV.2. Power Series Representations of Analytic Functions

Note. In this section, we finally prove the fact that an analytic function of a complex variable (that is, a continuously differentiable function of a complex variable) has a power series representation. For more on this idea, see the introductory note in Section III.2. Analytic Functions. Recall that Corollary III.2.9 states that a power series $\sum a_n(z-a)^n$ with radius of convergence $R > 0$ is analytic on the disk $B(a; R)$. The main result of this section is the converse of Corollary III.2.9, given in Theorem IV.2.8. We use geometric series and (surprisingly) integrals to establish our main result. We start with a result which one sees in Advanced Calculus (unfortunately, ETSU does not have a class on advanced calculus) as "Leibniz's Rule."

Proposition IV.2.1. Let $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ be a continuous function and define $g : [c,d] \to \mathbb{C}$ by $g(t) = \int_a^b \varphi(s,t)\,ds$. Then g is continuous. Moreover, if $\dfrac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a,b] \times [c,d]$ then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t)\,ds.$$

Note. We need the following to prove our main result. It is justified by a straightforward computation.

Lemma IV.2.A. If $|z| < 1$ then $\displaystyle\int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\,ds = 2\pi.$

Note. The text calls the following result "transitory." However, it will lead us from "continuously differentiable" to "power series."

Proposition IV.2.6. Let $f : G \to \mathbb{C}$ be analytic and suppose $\overline{B}(a;r) \subseteq G$ $(r > 0)$. If $\gamma(t) = a + re^{it}$, and $0 \le t \le 2\pi$. Then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

for $|z-a| < r$.

Note IV.2.A. We use Proposition IV.2.6 to introduce series as follows:

$$\frac{1}{w-z} = \frac{1}{w-a}\frac{1}{1-\frac{z-a}{w-a}} = \frac{1}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n.$$

Since $|z-a| < r = |w-a|$, the series converges absolutely. We then get

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$$

and with $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$ we have by Proposition IV.2.6 that

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n\right)dw$$

$$= \frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)}{w-a}\lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{z-a}{w-a}\right)^n\right)dw =^* \lim_{N\to\infty}\left(\frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)}{w-a}\sum_{n=0}^{N}\left(\frac{z-a}{w-a}\right)^n\right)dw\right)$$

$$= \lim_{N\to\infty}\sum_{n=0}^{N}\left[\frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)}{w-a}\left(\frac{z-a}{w-a}\right)^n\right)dw\right] = \sum_{n=0}^{\infty}\left[\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw\right)(z-a)^n\right],$$

provided that the equality in red* holds. This requires uniform convergence of the sequence of functions.

Note IV.2.B. We addressed uniform convergence in Section II.6. Uniform Convergence. Recall that a sequence of Riemann integrable functions $\{f_n\}$ which converges uniformly to function f on $[a,b]$, the Riemann integral of f equals the limit of the Riemann integrals of f_n:

$$\int_a^b f(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right).$$

See my online notes for Analysis 2 (MATH 4227/5227) on Section 8.1. Sequences of Functions (notice Theorem 8-3) for details. The next lemma shows that a similar result holds for line integrals.

Lemma IV.2.7. Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n and F are continuous on $\{\gamma\}$. If F is the uniform limit of F_n on $\{\gamma\}$ then

$$\int_\gamma F = \int_\gamma (\lim F_n) = \lim\left(\int_\gamma F_n\right).$$

Note. Now, we FINALLY resolve our definition of analytic with that used by the real analysts.

Theorem IV.2.8. Let f be analytic in $B(a;R)$. Then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a| < R$ where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence $\geq R$.

Note. Several corollaries follow from Theorem IV.2.8. They relate analytic f to properties of the power series representing it. In particular, Corollary IV.2.13 is useful since it relates the value of a particular integral to the value of a particular derivative (think: integrals are *hard* to evaluate and derivatives are *easy* to find and evaluate, making the given relationship computationally useful).

Corollary IV.2.11. If $f : G \to \mathbb{C}$ is analytic and $a \in G$ then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z - a| < R$ where $R = d(a, \partial G)$.

Corollary IV.2.12. If $f : G \to \mathbb{C}$ is analytic then f is infinitely differentiable.

Corollary IV.2.13. If $f : G \to \mathbb{C}$ is analytic and $\overline{B}(a; r) \subseteq G$ then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw$$

where $\gamma(t) = a + re^{it}$ and $t \in [0, 2\pi]$.

Note. We can use Corollary 2.13 to evaluate integrals using derivatives.

Exercise IV.2.7(b). Evaluate $\displaystyle\int_\gamma \frac{dz}{z-a}$ where $\gamma(t) = a + re^{it}$.

Solution. We use Corollary 2.13 with $f(z) = 1$ (which is analytic in all of $\mathbb{C}$, in particular it is analytic on $\overline{B}(a; r)$) and $n = 0$. Then

$$f^{(0)}(a) = 1 = \frac{0!}{2\pi i} \int_\gamma \frac{1}{w-a}\, dw \text{ or } \int_\gamma \frac{1}{w-a}\, dw = 2\pi i.$$

That is, the given integral has the value $2\pi i$. Notice that this reduces to Example IV.1.A of Section IV.1. Riemann-Stieltjes Integrals if we take $a = 0$ and $r = 1$. $\square$

Note. The following is an integral step in our proof of the Fundamental Theorem of Algebra, as we'll see in Section IV.3. Zeros of an Analytic Function (see Theorem IV.3.4, Liouville's Theorem, and Theorem IV.3.5, Fundamental Theorem of Algebra). It allows us to put a bound on $|f^{(n)}(a)|$ for an analytic function f.

Theorem IV.2.14. Cauchy's Estimate. Let f be analytic in $B(a;R)$ and suppose $|f(z)| \le M$ for all $z \in B(a;R)$. Then

$$|f^{(n)}(a)| \le \frac{n!M}{R^n}.$$

Note IV.2.C. The next proposition shows that an analytic function f on $B(a;R)$ has a primitive (i.e., an antiderivative) on $B(a;R)$. As shown in the proof, a primitive of f can be found by integrating the power series of f term-by-term. That is,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \text{ on } B(a;R) \text{ and } F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1} \text{ on } B(a;R)$$

implies $F'(z) = f(z)$ on $B(a;R)$.

Proposition IV.2.15. Let f be analytic in $B(a;R)$ and suppose γ is a closed rectifiable curve in $B(a;R)$. Then f has a primitive in $B(a;R)$ and so $\int_\gamma f = 0$.

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