IV.4. The Index of a Closed Curve

Note. In this section, we consider a geometric property of closed rectifiable curves. We use integrals to define the *winding number* of such a curve γ about a point a not on the curve. We will relate the winding number of γ to integrals over γ for certain functions in Section 5. Cauchy's Theorem and Integral Formula; see Cauchy's Integral Formula (First and Second Versions), Theorems IV.5.4 and IV.5.6 respectively. The results of this section are foreshadowed by the integral

$$\int_\gamma \frac{1}{z-a}\,dz = 2\pi i n,$$

where $\gamma(t) = a + re^{int}$ for $t \in [0, 2\pi]$. We computed this integral in Example IV.1.A of Section IV.1. Riemann-Stieltjes Integrals in the special case that $a = 0$ and $n = 1$. We start with a result concerning the same integrand $1/(z-a)$, but for an arbitrary closed rectifiable curve γ not containing a (notice that the *value* of the integral is not given, but instead a *property* of the integral).

Proposition 4.1. If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$$

is an integer.

Definition. If γ is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$,

$$n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$$

is the *winding number* of γ around a (or *index* of γ with respect to a).

Definition. Recall that for $\gamma : [0,1] \to \mathbb{C}$, we have $-\gamma = \gamma^{-1}$ is defined as $\gamma(1-t)$ (see Section IV.1. Riemann-Stieltjes Integrals). If also $\sigma : [0,1] \to \mathbb{C}$ and $\gamma(1) = \sigma(0)$ then $\gamma + \sigma : [0,1] \to \mathbb{C}$ is defined as

$$\gamma + \sigma = \begin{cases} \gamma(2t), & t \in [0,1/2] \\ \sigma(2t-1), & t \in [1/2,1]. \end{cases}$$

Proposition IV.4.3. If γ and σ are closed rectifiable curves having the same initial points then

(a) $n(\gamma; a) = -n(-\gamma; a)$ for all $a \notin \{\gamma\}$, and

(b) $n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a)$ for all $a \notin \{\gamma\} \cup \{\sigma\}$.

Note. The proof of Proposition IV.4.3 is to be given in Exercise IV.4.1. The idea of the winding number is that it gives, when positive, the number of times γ goes around a in the positive (counter clockwise) direction. If the winding number is negative, then it gives in absolute value the number of times γ goes around a in the negative (clockwise) direction. Conway gives, on his page 82, an informal argument that integrals of $1/(z-a)$, were they evaluated using antiderivatives (which would have to be some branch of the logarithm, $\log(z-a)$, but such a branch cannot be defined on γ because of the branch cut, as argued in Note III.2.D of Section III.2. Analytic Functions) would give integral multiples of $2\pi i$.

Note. Recall that a set X in $\mathbb{C}$ is *connected* if the only subsets of X which are both open and closed with respect to X are $\varnothing$ and X (see the definition of *connected* in

Section II.2. Connectedness). A subset $D \subseteq X$ in $\mathbb{C}$ is a *component* of X if it is a maximal connected subset of X. That is, D is connected and there is no connected subset of X that properly contains D (see Definition II.2.5, also in Section II.2. Connectedness).

Note. For γ a closed rectifiable curve in $\mathbb{C}$, we have that the set $G = \mathbb{C} \setminus \{\gamma\}$ is an open set. Also, since $\{\gamma\}$ is a compact set (it is a continuous image of the compact set $[0, 1]$, and so is compact by Theorem II.5.8(a)), then by the Heine-Borel Theorem (Theorem II.4.10), $\{\gamma\}$ is closed and bounded so that it is a subset of $B(0; R)$ for some R sufficiently large. Then $\{z \in \mathbb{C} \mid |a| > R\} \subset G$, so that G has one and only one unbounded component. The next result addresses, in part, the winding number of γ about the points in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$.

Theorem IV.4.4. Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G.

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