IV.6. The Homotopic Version of Cauchy's Theorem and Simple Connectivity

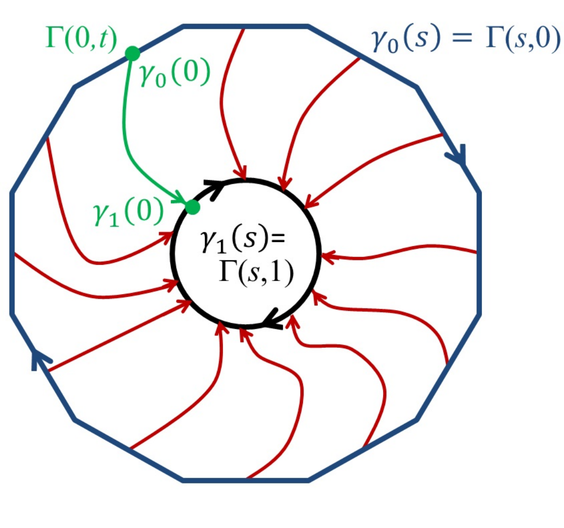
Note. In this section, we give a more thorough version of Cauchy's Theorem. Informally, we show that if path γ_0 can be continuously transformed into path γ_1 over the region of analyticity of function f, then $\int_{\gamma_0} f = \int_{\gamma_1} f$. This is accomplished in Cauchy's Theorem, Third Version (Theorem IV.6.7). The process of continuously transforming is accomplished through a "homotopy."

Definition IV.6.1. Let $\gamma_0, \gamma_1 : [0,1] \to G$ be two closed rectifiable curves in a region G. Then γ_0 is *homotopic* to γ_1 in G if there is a continuous function $\Gamma : [0,1] \times [0,1] \to G$ such that

$$\begin{cases} \Gamma(s,0) = \gamma_0(s) \text{ and } \Gamma(s,1) = \gamma_1(s) \text{ for } s \in [0,1] \\ \Gamma(0,t) = \Gamma(1,t) \text{ for } t \in [0,1]. \end{cases}$$

Γ is a *homotopy*, and this is denoted $\gamma_0 \sim \gamma_1$ (where G is understood).

Note. Closed rectifiable paths γ_0 and γ_1,and homotopy Γ are related as follows:



Note. We have that for each $t \in [0,1]$, $\gamma_t(s) = \Gamma(s,t)$ is closed. However, we do not require $\gamma_t(s)$ to be rectifiable, though this will, in practice, be the case.

Note IV.6.A. It is fairly easy to show that "homotopic," $\sim$, is an equivalence relation. We have that $\sim$ is reflexive and $\gamma \sim \gamma$, as is shown by homotopy $\Gamma(s,t) = \gamma(s)$. If $\gamma_0 \sim \gamma_1$ under homotopy $\Gamma(s,t)$, then $\gamma_1 \sim \gamma_0$ as is shown by homotopy $\Lambda(s,t) = \Gamma(s, 1-t)$. Hence $\sim$ is symmetric. Suppose $\gamma_0 \sum \gamma_1$ and $\gamma_1 \sim \gamma_2$, say under homotopies $\Gamma(s,t)$ and $\Lambda(s,t)$ respectively. Then

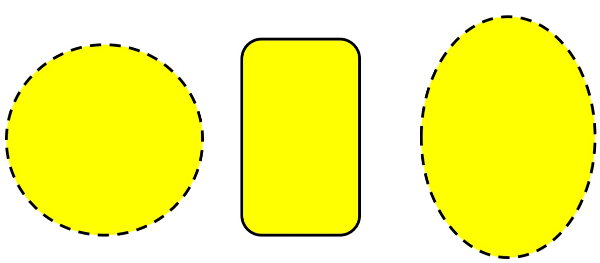
$$\Phi(s,t) = \begin{cases} \Gamma(s,2t) & \text{for } 0 \le t \le 1/2 \\ \Lambda(s, 2t-1) & \text{for } 1/2 < t \le 1 \end{cases}$$

is a homotopy between γ_0 and γ_1, so that $\sim$ is transitive. Therefore, $\sim$ is an equivalence relation.

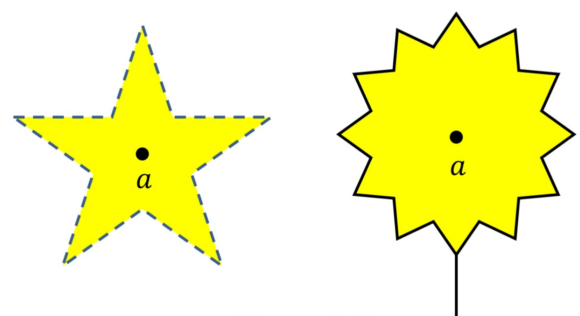
Note IV.6.B. In Section IV.4. The Index of a Closed Curve we saw how to find the inverse of a path γ (we denoted the inverse as $-\gamma$) and how to add paths γ and σ where $\gamma(1) = \sigma(0)$ (denoted $\gamma + \sigma$). We now give a brief description of the use of these ideas in the area of algebraic topology. This information can be found in Appendix A of Andrew Wallace's *Algebraic Topology: Homology and Cohomology* (NY: W. A. Benjamin, 1970). This book is still in print through Dover Publications, but and can also be viewed online on the archive.org (accessed 12/6/2023). We can form a group out of the equivalence classes of closed paths "based" at some point $x \in E$. It can also be shown that the set of equivalence classes (called *homotopy classes*) of closed paths based at x form a group under the operation $+$ given in our Section IV.4 (Theorem A-6). This group is called the *fundamental group of E* *with respect to the base point* x and is denoted $\pi(E, x)$ (Definition A-9). If points $x, y \in E$ can be joined by a path in E, then $\pi(E, x) \cong \pi(E, y)$ (Theorem A-7). In fact, the fundamental group of a space E is a topological invariant of the space; that is, if spaces E and F are homeomorphic (i.e., there is a one to one and onto continuous mapping from E to F) then the fundamental group of E is isomorphic to the fundamental group of F (Exercise A-4).

Definition. A set G is *convex* if given any two points a and b in G, the line segment joining a and b, $[a, b]$, lies entirely in G. The set G is *star shaped* if there is a point a in G such that for each $z \in G$, the line segment $[a, z]$ lies entirely in G. Such a set is *a-star shaped*.

Note. Some examples of convex and a-star shaped sets are the following:



Convex Sets



a-star Shaped Sets

Proposition IV.6.4. Let G be an open set which is a-star shaped. If γ_0 is the curve which is constantly equal to a (that is, $\gamma_0(t) = a$ for $t \in [0, 1]$), then every closed rectifiable curve in G is homotopic to γ_0.

Definition. If γ is a closed rectifiable curve in G then γ is *homotopic to zero* ($\gamma \sim 0$) if γ is homotopic to a constant curve.

Note. The equivalence class of all curves homotopic to zero form the identity element in the fundamental group of G (hence the terminology).

Note. We now link Cauchy's Theorem to curve homotopy. The Second Version of Cauchy's Theorem is a special case of the Third Version. We offer a proof of the Third Version, so we skip a proof of the Second Version.

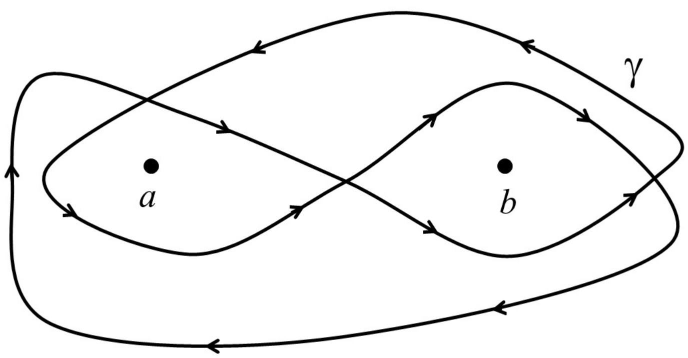
Theorem IV.6.6. Cauchy's Theorem (Second Version).
If $f : G \to \mathbb{C}$ is an analytic function and γ is a closed rectifiable curve in G such that $\gamma \sim 0$, then $\int_\gamma f = 0$.

Theorem IV.6.7. Cauchy's Theorem (Third Version).
If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic on G.

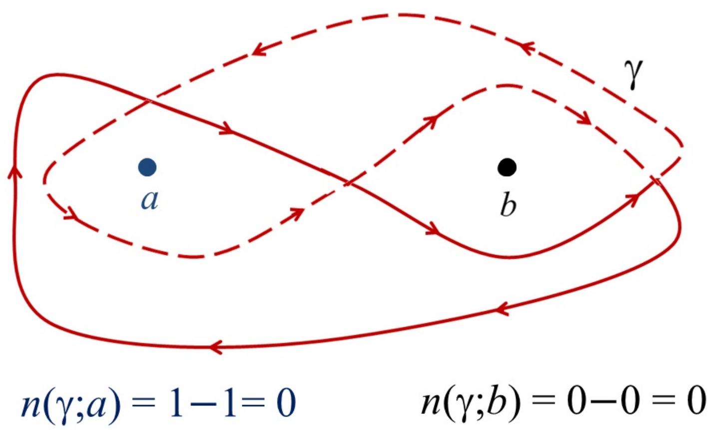
Note. The big idea in the proof of Cauchy's Theorem—Third Version is to get integrals over little quadrilaterals that lie inside small disks which are subsets of G, then to use Proposition IV.2.15. Now for the lengthy proof of Theorem IV.6.7.

Corollary IV.6.10. If γ is a closed rectifiable curve in G such that $\gamma \sim 0$, then $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$.

Note. The converse of Corollary 6.10 does not hold. Consider, for example, Problem IV.6.8: Let $G = \mathbb{C} \setminus \{a, b\}$, $a \neq b$, and γ:



Then for all $w \in \mathbb{C} \setminus G$ (which is just $w = a$ and $w = b$) we have $n(\gamma; a) = n(\gamma; b) = 0$:

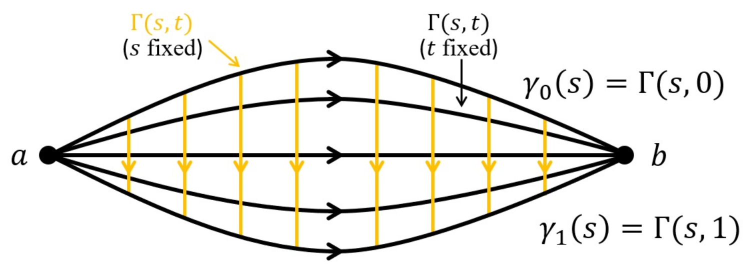


However $\gamma \not\sim 0$ ("convince yourself" as the text says—imagine nails at a and b and γ as a loop of string).

Note. Next we consider homotopy of non-closed rectifiable curves.

Definition. If $\gamma_0, \gamma_1 : [0,1] \to G$ are two rectifiable curves in G such that $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$. Then γ_0 and γ_1 are *fixed-end-point homotopic* ("FEP" homotopic) if there is a continuous map $\Gamma : [0,1] \times [0,1] \to G$ such that

$$\begin{cases} \Gamma(s,0) = \gamma_0(s) \text{ and } \Gamma(s,1) = \gamma_1(s) \text{ for } s \in [0,1] \\ \Gamma(0,t) = a \text{ and } \Gamma(1,t) = b \text{ for } t \in [0,1]. \end{cases}$$



Note. For two given points $a, b \in G$, the relation of fixed-end-point homotopic between curves with a and b as their initial and final end points (respectively) is an equivalence relation, is as to be shown in Exercise IV.6.3. If γ_0 and γ_1 are rectifiable curves from a to b, then $\gamma_0 + (-\gamma_1) = \gamma_0 = \gamma_1$ is a closed rectifiable curve. Suppose γ_0 and γ_1 are fixed-end-point homotopic, and let Γ be the homotopy. Define $\gamma : [0,1] \to G$ as

$$\gamma(s) = \begin{cases} \gamma_0(3s) & \text{for } 0 \leq s \leq 1/3 \\ b & \text{for } 1/3 < s < 2/3 \\ -\gamma_1(3-3s) & \text{for } 2/3 \leq s \leq 1. \end{cases}$$

Notice that γ starts at a, traces out γ_0 to b, "sits" at b (for $1/3 \leq s \leq 2/3$), and then traces out $-\gamma_1$ back to a. We now show that $\gamma \sum 0$. Define $\Lambda : I^2 \to G$ as

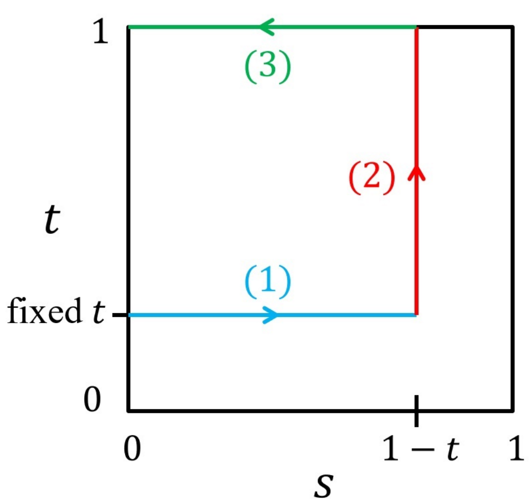
$$\Lambda(s,t) = \begin{cases} \Gamma(3s(1-t),t) & \text{for } 0 \le s \le 1/3 \\ \Gamma(1-t, 3s-1+2t-3st) & \text{for } 1/3 < s < 2/3 \\ \gamma_1((3-3s)(1-t)) & \text{for } 2/3 \le s \le 1. \end{cases}$$

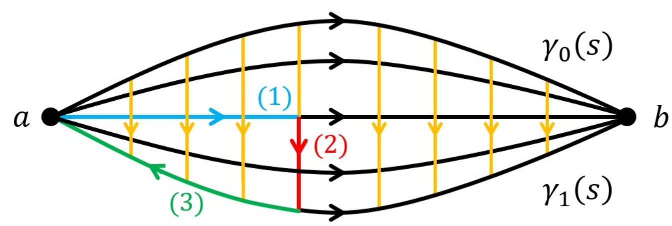
Since

$$\Gamma(3s(1-t),t)|_{s=1/3} = \Gamma(1-t,t) = \gamma)1-t, 3s-1+2t-3st)|_{s=1/3},$$

$$\Gamma(1-t, 3s-1+2t-3st)|_{s=2/3} = \Gamma(1-t,1) = -\gamma_1(t) = \gamma_1((3-3s)(1-t))|_{s=2/3},$$

and Γ and γ_1 are continuous, then $\Lambda(s,t)$ is continuous on I^2. Notice that for a fixed value of t, the first piece of Λ takes on the same values as $\Gamma(s,t)$ for s ranging from 0 to $1-t$, the second piece of Λ takes on the same values as $\Gamma(1-t,t')$ for t' ranging from t to 1, and the third piece of Λ takes on the same values as $-\gamma_1(t')$ for t' ranging from $1-t$ to 0. See the figure below.





Now $\Lambda(0,t) = \Gamma(0,t) = a$ and $\Lambda(1,t) = -\gamma_1(0) = \gamma_1(1) = b$. Also

$$\begin{aligned}
\Lambda(s,0) &= \begin{cases} \Gamma(3s,0) & \text{for } 0 \le s \le 1/3 \\ \Gamma(1,3s-1) & \text{for } 1/3 < s < 2/3 \\ \gamma_1(3-3s) & \text{for } 2/3 \le s \le 1 \end{cases} \\
&= \begin{cases} \gamma_0(3s) & \text{for } 0 \le s \le 1/3 \\ b & \text{for } 1/3 < s < 2/3 \\ \gamma_1(3-3s) & \text{for } 2/3 \le s \le 1 \end{cases} \\
&= \gamma(s),
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(s,1) &= \begin{cases} \Gamma(0,t) & \text{for } 0 \le s \le 1/3 \\ \Gamma(0,1) & \text{for } 1/3 < s < 2/3 \\ \gamma_1(0) & \text{for } 2/3 \le s \le 1. \end{cases} \\
&= \begin{cases} a & \text{for } 0 \le s \le 1/3 \\ a & \text{for } 1/3 < s < 2/3 \\ a & \text{for } 2/3 \le s \le 1 \end{cases} \\
&= a.
\end{aligned}$$

So by definition, γ is a constant curve (i.e., $\gamma(t) = a$ for all $t \in [0,1]$) and $\gamma \sim 0$. By Cauchy's Theorem Second Version (Theorem 6.6) we have $\int_\gamma f = \int_{\gamma_0 - \gamma_1} f = 0$, from which we get the next theorem.

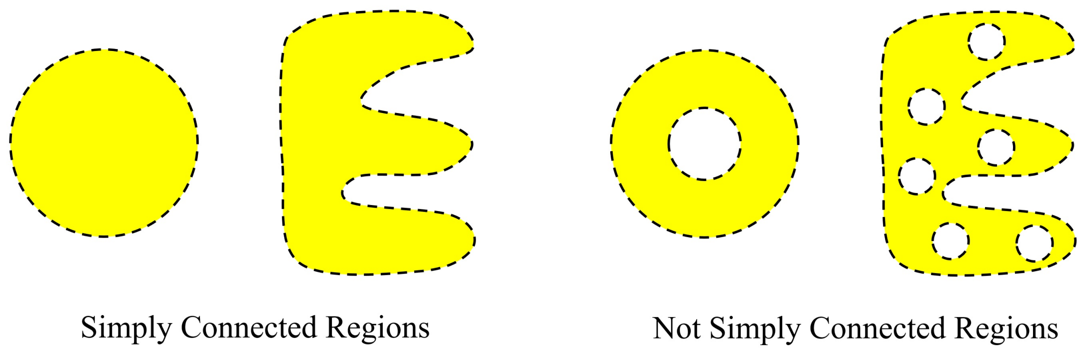
Theorem IV.6.13. Independence of Path Theorem.

If γ_0 and γ_1 are two rectifiable curves in G from a and b and γ_0 and γ_1 are fixed-end-point homotopic then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for any function f analytic in G.

Note. The following definition is really a topological concept.

Definition. An open set G is *simply connected* if G is connected and every closed curve G is homotopic to zero.

Note. A region is simply connected if it is connected and has "no holes":



Note. We have already seen a region that is not simple in connection with Cauchy's Theorem (First Version). See Note IV-5-A. In that case we required that simple closed curves γ_k have winding numbers which sum to 0 got all points outside of region G on which function f is analytic (i.e., $\sum_{k=1}^{m} N(\gamma_k; w) = 0$ for all $x \in \mathbb{C} \setminus G$). This insured that the sum of the integrals over the γ_k of f was 0. In the next version of Cauchy's Theorem, we consider only one closed rectifiable curve, but we have region G as a simply connected so that the winding number of γ about all points outside of G is 0. Predictably, this implies that the integral of f over γ is 0.

Theorem IV.6.15. Cauchy's Theorem (Fourth Version).

If G is simply connected then $\int_\gamma f = 0$ for every closed rectifiable curve and every analytic function f on G.

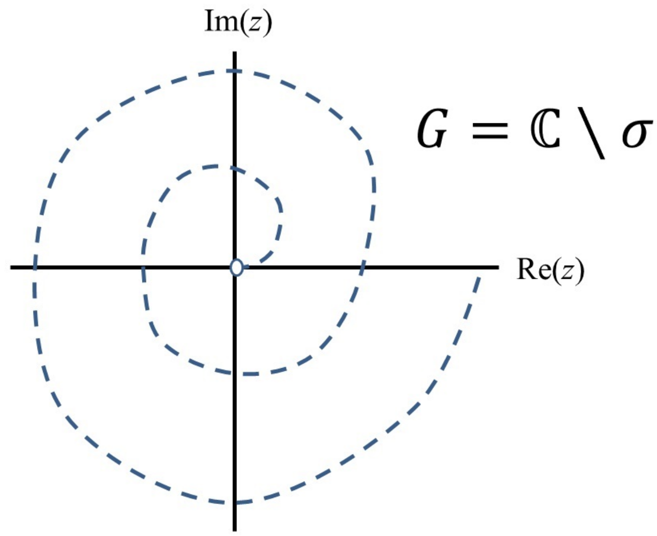
Note. The following brings primitives back into the picture. Recall that $\int_\gamma f = 0$ if γ is closed and f is continuous with a primitive on G (by Theorem IV.1.18, our Fundamental Theorem of Calculus, or by Corollary IV.1.22).

Corollary IV.6.16. If open G is simply connected and $f : G \to \mathbb{C}$ is analytic in G then f has a primitive in G.

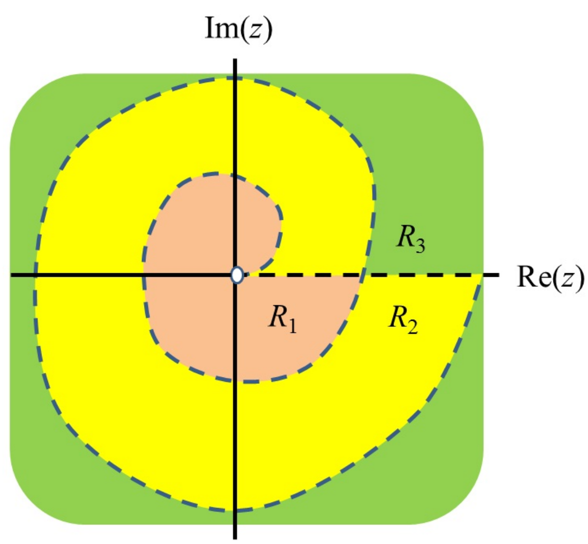
Note. The Next result is similar to Corollary 6.16, but deals with branches of the logarithm.

Corollary IV.6.17. Let G be simply connected and let $f : G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any $z \in G$. Then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \exp(g(z))$ (i.e., g is a branch of $\log(f(z))$ on G). If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.

Note. Corollary 6.17 verifies some observations we have made about branches of the logarithm. For example, let $\sigma = \{z \mid z = te^{it}, t \in [0, \infty)\}$:



Then G is simply connected, $f(z) = z$ is nonzero on G, and so there is a branch of the logarithm on G. One such branch can be defined as follows. Consider a partition of $\mathbb{C} \setminus \sigma$ into regions:



For $z \in R_n$ define $\log(z) = \log|z| + \theta i$ where θ is an argument of z in $(2\pi(n-1), 2\pi n]$.