IV.7. Counting Zeros; The Open Mapping Theorem

Note. We now use Cauchy's Integral Theorem to (1) count the number of zeros of f inside a curve, and (2) prove that a nonconstant analytic function maps open sets to open sets (WHOA!).

Theorem IV.7.2. Let G be a region and let f be an analytic function on G with zeros $a_1, a_2, \ldots, a_m$ (repeated according to multiplicity). If γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma \sim 0$ in G, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^m n(\gamma; a_k).$$

Corollary IV.7.3. Let f, G, and γ be as described in Theorem 7.2, except that $a_1, a_2, \ldots, a_m$ are the points in G such that $f(z) = \alpha$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz = \sum_{k=1}^m n(\gamma; a_k).$$

Example IV.7.A. Evaluate $\displaystyle\int_\gamma \frac{2z+1}{z^2+z+1}\,dz$ where $\gamma(t) = 2e^{it}$ for $t \in [0, 2\pi]$.

Definition. A *simple root* of the equation $f(z) = \zeta$ is a zero of function $f(z) - \zeta$ of multiplicity 1.

Note. Exercise IV.7.3 says: "Let f be analytic in $B(a;R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m if and only if $f^{(m-1)}(a) = f^{(m-2)}(a) = \cdots = f'(a) = f(a) = 0$ and $f^{(m)}(a) \neq 0$." So a is a simple root of $f(z) = 0$ if $f(a) = 0$ and $f'(a) \neq 0$.

Theorem IV.7.4. The Stability Theorem for Orders of Zeros of Equations.

Suppose f is analytic in $B(a;R)$ and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there is an $\varepsilon > 0$ and $\delta > 0$ such that for $0 < |\zeta - \alpha| < \delta$ the equation $f(z) = \zeta$ has exactly m simple roots in $B(a;\varepsilon)$.

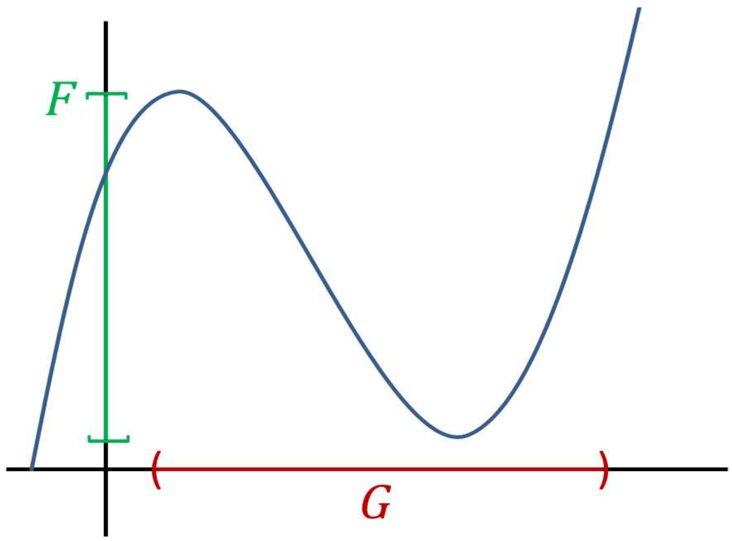
Note. The proof of Theorem IV.7.4 is given in a, sort of, self-contained supplement which describes the concept of stability.

Note. We see from Theorem 7.4 that any ζ with $|\zeta - \alpha| < \delta$ (i.e., $\zeta \in B(\alpha;\delta)$), there is a solution to $f(z) = \zeta$ in $B(a;\varepsilon)$. So $f(B(a;\varepsilon)) \supseteq B(\alpha;\delta)$. This observation will be used to prove the Open Mapping Theorem.

Theorem IV.7.5. The Open Mapping Theorem.

Let G be a region (open connected set) and suppose that f is a nonconstant analytic function on G. Then for any open set U in G, $F(U)$ is open.

Note. Of course, there is no parallel of the Open Mapping Theorem in the real setting. A simple cubic polynomial of a real variable can map an open interval to a closed interval:



Corollary IV.7.6. Suppose $f : G \to \mathbb{C}$ is one to one, analytic and $f(G) = \Omega$. Then $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $(f^{-1})'(\omega) = 1/f'(z)$ where $\omega = f(z)$.

Note. A somewhat surprising implication of the Open Mapping Theorem is given in Exercise IV.7.4: "Suppose that $f : G \to \mathbb{C}$ is analytic and one-one: show that $f'(z) \neq 0$ for any z in G." Again, this is not valid in the real setting as we can see by considering $f(x) = x^3$ in an open interval containing $x = 0$.

Revised: 2/12/2018