IX.2. Analytic Continuation Along a Path.

Note. Consider a function f analytic on a region G. If f_1 is a function analytic on a region G_1 such that G_1 properly contains G and $f(z) = f_z(z)$ for all $z \in G$, then f_1 is an "analytic continuation" of f from G to G_1. In Section VII.8, the Riemann zeta function is defined on $\text{Re}(z) > 1$ and then extended to $\text{Re}(z) > 0$, $z \neq 1$, to $\text{Re}(z) > -1$, $z \neq 1$, and finally to $\mathbb{C} \setminus \{1\}$. This is an example of analytic continuation.

Note. Another example of analytic continuation is given by $g(z) = \sum_{n=1}^{\infty}(-1)^{n-1}z^n/n$ for $|z| < 1$ and $f(z) = \log(1 - z)$ (the principal branch of the logarithm) for $\text{Re}(z) > -1$. Similar examples are given by different power series representations for the same analytic functions. For example, with $f(z) = 1/z$ the power series for f centered at 1, denote it f_1, is valid for $|z - 1| < 1$ and the power series for f centered at 2, denote it f_2, is valid for $|z - 2| < 2$. So f is an analytic continuation of f_2, and f_2 is an analytic continuation of f_1. We revise this verbiage some in Definition IX.2.2

Definition IX.2.1. A *function element* is a pair (f, G) where G is a region and f is an analytic function on G. For a given function element (f, G), the *germ of f at a* is the collection of all functions elements (g, D) such that $a \in D$ and $f(z) = g(z)$ for all z in a neighborhood of a, denoted $[f]_z$.

Note. Notice that if $(g, D) \in [f]_a$ where f is defined on a region G containing a, then there is a neighborhood of a such that $f(z) = g(z)$ for all z in the neighborhood. So we also have $(f, G) \in [g]_z$ (and conversely).

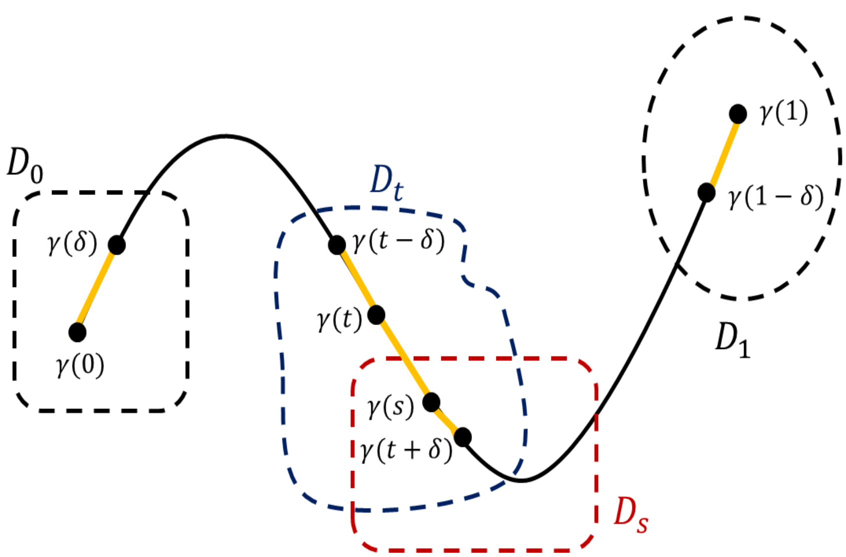
Definition IX.2.2. Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and suppose that for each $t \in [0, 1]$ there is a function element (f_t, D_t) such that

(a) $\gamma(t) \in D_t$;

(b) for each $t \in [0, 1]$ there is a $\delta > 0$ such that $|s - t| < \delta$ implies $\gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$.

Then $f_1, D_1)$ is the *analytic continuation of* (f_0, D_0) *along the path* γ, or "(f_1, D_1) is obtained from (f_0, D_0) by *analytic continuation* along γ."

Note. Schematically, Definition IX.2.2 gives:



Definition/Note IX.2.A. Notice that since $\{\gamma(s) \mid |s-t| < \delta\}$ is a connected set, then there is a connected component of $D_s \cap D_t$ which contains $\gamma(t)$ and $\gamma(s)$. In the definition of "germ of f at a" we have equality of functions for z in "a neighborhood" of a; by convention we take the D_s and D_t sufficiently small so that said neighborhood is the connected component of $D_s \cap D_t$ which contains $\gamma(s)$. (Conway comments on page 214: "Observe that the sets D_t in the definition can be enlarged or diminished without affecting the fact that there is a continuation.") Then $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$ for $|s-t| < \delta$ implies that $f_s(z) = f_t(z)$ for z in the component of $D_s \cap D_t$ that contains $\gamma(s)$.

Note. Conway states (page 214): "Whether for a given curve and a given function element there is an analytic continuation along the curve can be a difficult question." As a result, we give no existence theorems about analytic continuations. However, we state a uniqueness theorem, the Monodromy Theorem (theorem IX.3.6), in the next section which gives a condition when the analytic continuation along two different paths to the same point yields the same function (keep in mind extensions of a branch of the logarithm along paths about 0).

Theorem IX.2.4. Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_y) \mid 0 \leq t \leq 1\}$ and $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ be analytic continuations along given path γ such that $[f_0]_a = [g_0]_a$. Then $[f_1]_a = [g_1]_a$.

Note. The equality of $[f_1]_b$ and $[g_1]_b$ in Proposition IX.2.4 gives a uniqueness along a given path of "germ of f_1 at b." So we have the following definition.

Definition IX.2.6. If $\gamma : [0,1] \to \mathbb{C}$ is a path from a to b and $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ is an analytic continuation along γ then the *germ* $[f_1]_b$ *is the analytic continuation* of $[f_0]_a$ along path γ.

Note. The next definition concerns maximizing the domain through the process of analytic continuation.

Definition IX.2.7. If (f, G) is a function element then the *complete analytic function obtained from* (f, G) is the collection $\mathcal{F}$ of all germs $[g]_b$ for which there is a point $a \in G$ and a path from a to $b \in \mathbb{C}$ such that $[g]_b$ is the analytic continuation of $[f]_a$ along γ. A collection of germs $\mathcal{F}$ is a *complete analytic function* if there is a function element (f, G) such that $\mathcal{F}$ is the complete analytic function obtained from (f, G).

Note. The complete analytic function $\mathcal{F}$ above is a collection of germs (of functions) more than a function itself. We define $\mathcal{R} = \{(z, [f]_z) \mid [f]_z \in \mathcal{F}\}$ and then define $\mathcal{F} : \mathcal{R} \to \mathbb{C}$ as $\mathcal{F}((z, [f]_z)) = f(z)$. However, we need more structure on the domain $\mathcal{R}$ of $\mathcal{F}$ in order to discuss analyticity. This is accomplished in Section IX.5 ("The Sheaf of Germs of Analytic Functions on an Open Set") when we define the *Riemann Surface* of $\mathcal{F}$ in Definition IX.5.14.