IX.3. Monodromy Theorem.

Note. In Exercise IX.2.2, analytic continuation of $f_0 = g_0$, the principal branch of $\sqrt{2}$, is considered. For $D_0 = \{z \mid |z-1| < 1\}$, the analytic continuation of (f_0, D_0) along path $\gamma(t) = \exp(2\pi i t)$ (starting at $a = 1$ and ending at $b = 1$) is shown to satisfy $[f_1]_1 = [-f_0]_1$. The analytic continuation of $(f_0, D_0) = (g_0, D_0)$ along the path $\sigma(t) = \exp(4\pi i t)$ (starting at $a = 1$ and ending at $b = 1$) is shown to satisfy $[g_1]_1 = [g_0]_1$. So even though $[f_0]_1 = [g_0]_1$ and γ and σ are paths starting and ending at the same points, we have $[f_1]_b \neq [g_1]_b$. So there is some kind of dependence of the analytic continuation on the path. The Monodromy Theorem (Theorem IX.3.6) gives a condition involving fixed end point homotopy under which analytic continuation is path independence.

Note. Before stating and proving the Monodromy Theorem, we need two lemmas and a definition. The first lemma concerns radius of convergence.

Lemma IX.3.1. Let $\gamma : [0,1] \to \mathbb{C}$ be a point and let $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ be an analytic continuation along γ. For $0 \leq t \leq 1$ let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then either $R(T) = \infty$ or $T : [0,1] \to (0, \infty)$ is continuous.

Lemma IX.3.2. Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ be an analytic continuation along path γ. There is a number $\varepsilon > 0$ such that if $\sigma : [0,1] \to \mathbb{C}$ is any path from a to b with $|\gamma(t) - \sigma(t)| < \varepsilon$ for all $t \in [0,1]$, and if $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ is any continuation along σ with $[g_0]_a = [f_0]_a$; then $[g_1]_a = [f_1]_b$.

Definition IX.3.5. Let (f, D) be a function element and let G be a region which contains D. Then (f, D) *admits unrestricted analytic continuation in* G if for any path γ in G with initial point in D there is an analytic continuation of (f, D) along γ.

Note. A function element which admits unrestricted analytic continuation on a region still may have a continuation which is path dependent (see the note at the beginning of this section of notes). The Monodromy Theorem gives conditions under which the analytic continuations are path independent.

Theorem IX.3.6. Monodromy Theorem.
Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G. Let $a \in D$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b. Let $\{(f_t, D_t) \mid 0 \leq t \leq 1\}$ and $\{(g_t, B_t) \mid 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed end point homotopic in G then $[f_1]_b = [g_1]_b$.

Note. The following is a simpler version concerning the uniqueness of the Monodromy Theorem.

Corollary IX.3.9. Let (f, D) be a function element which admits unrestricted continuation in the simply connected region G. Then there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all $z \in D$.

Note. Once again, we consider the branch of the logarithm on some region which does not contain 0 (or, as we did at the beginning of this section, a branch of $\sqrt{z}$). We know that a branch of the logarithm cannot be defined at 0, since e^z is never 0 by Lemma III.2.A. So any region for which we produce an analytic continuation of a branch of the logarithm must exclude 0. To apply Corollary IX.3.9, we must have a simply connected region. Any simply connected region which excludes 0 must also exclude some path from 0 "to ∞," which is the branch cut necessary to produce a larger branch of the logarithm.

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