IX.4. Topological Spaces and Neighborhood Systems.

Note. A topological space consists of a set X of points and a set $\mathcal{T}$ of subsets of X which are, by definition, open. In the formal definition of a topological space, we take our lead from the behavior of open sets as given in Proposition II.1.9. Many of the results will be familiar from the metric space setting and the proofs will often be similar. Most of the results of this section are covered in our Introduction to Topology (MATH 4357/5357) class. Notes from this class based on J. R. Munkres' *Topology*, 2nd Edition (Prentice Hall, 2000) are online at `http://faculty.etsu.edu/gardnerr/5357/notes.htm`. We omit proofs from most of this section and instead refer to the proofs either from the metric space setting or from the Introduction to Topology setting.

Definition IX.4.1. A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X having the following properties:

(a) $\varnothing \in \mathcal{T}$ an d$X \in \mathcal{T}$;

(b) if $U_1, U_2, \ldots, U_n$ are in $\mathcal{T}$ then $\cap_{j=1}^n U_j \in \mathcal{T}$;

(c) if $\{U_i \mid i \in I\}$ is any collection of sets in $\mathcal{T}$ then $\cup_{i \in I} U_i$ is in $\mathcal{T}$.

Note. Similar to the definition of "closed set" in the metric space setting (Definition II.1.10) we have the following.

Definition IX.4.2. A subset F of a topological space X is *closed* if $X \setminus F$ is open. A point $a \in X$ is a *limit point* of a set A if for every open set U that contains a there is a point x in $A \cap U$ such that $x \neq a$.

Note. The next result corresponds to Proposition II.1.11 and Theorem 17.1 of Munkres.

Proposition IX.4.3. Let $(X, \mathcal{T})$ be a topological space. Then:

(a) $\varnothing$ and X are closed sets;

(b) if $F_1, F_2, \ldots, F_n$ are closed sets then $F_1 \cup F_2 \cup \cdots \cup F_n$ is closed;

(c) if $\{F_i \mid i \in I\}$ is a collection of closed sets then $\cap_{i \in I} F_i$ ic a closed set.

Note. The nest result corresponds to Proposition II.1.13(b) and Corollary 17.7 of Munkres.

Proposition IX.4.4. A subset of a topological space is closed if and only if it contains all its limit points.

Note. Not all topological spaces have open sets which are the result of being open based on the ε-definition of open under a metric. Such spaces are said to be *nonmetrizable*. One way to show that a topological space is nonmetrizable is to show that its open sets violate the properties of open sets in a metric space. See Section 21 of Munkres, "The Metric Topology (continued)," for two examples of nonmetrizable spaces. Conway gives these nonmetrizable spaces on page 223, one of which we give here. By the way, a topological space for which there is a metric producing the open sets, is called *metrizable* and the topology is the *metric topology*. First we need a definition of limit in a topological space.

Definition. In topological space $(X, \mathcal{T})$, sequence $\{x_n\} \subset X$ has limit $x \in X$ if for all open sets U containing x, there is $N \in \mathbb{N}$ such that for $n \geq N$ we have $x_n \in U$.

Example. Let $X = [0, 1]$ and let $\mathcal{T}$ consist of all sets U such that:

(i) if $0 \in U$ then $X \setminus U$ is either empty or a sequence of points in X (i.e., $X \setminus U$ is either empty or countable);

(ii) if $0 \notin U$ then U can be any set.

"It is left to the reader to prove that $(X, \mathcal{T})$ is a topological space." ASSUME there is a metric d for which $\mathcal{T}$ is the metric topology (i.e., assume $(X, \mathcal{T})$ is metrizable). That is, $U \in \mathcal{T})$ if and only if for each $x \in U$ there is $\varepsilon > 0$ such that $B(x; \varepsilon) \subset U$. Let $A = (0, 1)$. If $U \in \mathcal{T}$ and $0 \in U$ then there is $a \in U \cap A$. So 0 is a limit point of A in $(X, \mathcal{T})$. Then (under the metric d) there is a sequence $\{t_n\} \subset A$ such that $d(t_n, 0) \to 0$. For $U = [0, 1] \setminus \{t_1, t_2, \ldots\}$ we have that $0 \in U$ and U is open (by (i) since $X \setminus U = \{t_1, t_2, \ldots\}$). But then for $t_n \to 0$ in $(X, \mathcal{T})$ we need $t_n \in U$ for all $n \geq N$ for some $N \in \mathbb{N}$. Since no $t_n \in U$, this CONTRADICTION shows that the assumption that $(X, \mathcal{T})$ is metrizable is false. This example is similar to Munkres' examples which are based on the Sequence Lemma (Munkres' Lemma 21.2) which states that in a metrizable space, if $t \in A^-$ then there is a sequence $\{t_n\} \subset A$ such that $t_n \to t$ in the topological space.

Note. You are familiar with limits of sequences and functions being unique when they exist. However, there are topological spaces where limits may not be unique. These are not the places we want to do analysis or applied math! We prefer spaces in which we can separate points. For a general discussion on this, see Munkres' Section 31, "The Separation Axioms." Here, we restrict our attention to one particular separation axiom.

Definition IX.4.5. A topological space $(X, \mathcal{T})$ is a *Hausdorff space* if for any two distinct points a and b in X there are disjoint open sets U and V such that $a \in U$ and $b \in V$. Such a space is also called T_2.

Note. The definition of "connected set" is the same as for the metric space setting (Definition II.2.1). This is the same as Munkres' definition (see Section 23, "Connected Sets") though the terminology is slightly different.

Definition IX.4.6. A topological space $(X, \mathcal{T})$ is *connected* if the only nonempty subset of X which is both open and closed is the set X itself.

Note. The following result corresponds to Lemma II.2.6 (see Munkres' Theorem 25.1).

Proposition IX.4.7. Let $(X, \mathcal{T})$ be a topological space. Then $X = \sup\{C_i \mid i \in I\}$ where each C_i is a component of X (a maximal connected subset of X). Furthermore, distinct components of X are disjoint and each component is closed.

Note. Since there is no metric necessarily present for a topological space, then we cannot use the usual ε/δ definition of limit of a function. Instead, we take our inspiration from Proposition II.5.3 (see Munkres' Section 18, "Continuous Functions").

Definition IX.4.8. Let $(X, \mathcal{T})$ and $(\Omega, \mathcal{S})$ be topological spaces. A function $f : X \to \Omega$ is *continuous* if $f^{-1}(\Delta) \in \mathcal{T}$ whenever $\Delta \in \mathcal{S}$.

Note. The next result expresses continuity in terms of closed sets. Part (c) is analogous to the ε/δ definition where " for all $\varepsilon > 0$" is replaced with "for all open sets. . . ."

Proposition IX.4.9. Let $(X, \mathcal{T})$ and $(\Omega, \mathcal{S})$ be topological spaces and let $f : X \to \Omega$ be a function. Then the following are equivalent:

(a) f is continuous;

(b) if Γ is a closed subset of Ω then $f^{-1}(\Gamma)$ is a closed subset of X;

(c) if $a \in X$ and if $f(a) \in \Delta \in \mathcal{S}$ then there is a set $U \in \mathcal{T}$ such that $a \in U$ and $f(U) \subset \Delta$.

Note. The next proposition justifies the use of the term "continuous" for functions satisfying Definition IX.4.8 and Proposition IX.4.9. A continuous function maps connected sets to connected sets; it doesn't "break" a continuous set (a "continuum"), it keeps it continuous. This is related to Theorem II.5.8(b) from the metric space setting (and is Munkres' Theorem 23.4).

Proposition IX.4.10. Let $(X, \mathcal{T})$ and $(\Omega, \mathcal{S})$ be topological spaces and suppose that X is connected. If $F : X \to \Omega$ is a continuous function such that $f(X) = \Omega$ then Ω is connected.

Note. Our definition os "compact set" is the usual one (the universal one, really) and we can show that continuous functions map compact sets to compact sets). Bo Proposition IX.4.12 is related to Theorem II.5.8(a) and is Munkres' Theorem 26.5.

Definition IX.4.11. A set $K \subset X$ is *compact* is for every sub-collection $\mathcal{O}$ of $\mathcal{T}$ such that $K \subset \cup\{U \mid U \in \mathcal{O}\}$ there are a finite number of sets $U_1, U_2, \ldots, U_n$ in $\mathcal{O}$ such that $K \subset \cup_{k=1}^n U_k$.

Proposition IX.4.12. Let $(X, \mathcal{T})$ and $(\Omega, \mathcal{S})$ be topological spaces and suppose K is a compact subset of X. If $f : X \to \Omega$ is a continuous function then $f(K)$ is compact in Ω.

Definition IX.4.13. If Y is a subset of a topological space $(X, \mathcal{T})$ then $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$ is the *relative topology of* Y. A subset W of Y is *relatively open in* Y if $W \in \mathcal{T}_Y$. W is *relatively closed in* Y if $Y \setminus W \in \mathcal{T}_Y$.

Proposition IX.4.14. Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X.

(a) If X is compact and Y is a closed subset of X then $(Y, \mathcal{T}_Y)$ is compact.

(b) Y is a compact subset of X if and only if $(Y, \mathcal{T}_Y)$ is a compact topological space.

(c) If $(X, \mathcal{T})$ is a Hausdorff space then $(Y, \mathcal{T}_Y)$ is a Hausdorff space.

(d) If $(X, \mathcal{T})$ is a Hausdorff space and $(Y, \mathcal{T}_Y)$ is compact then Y is a closed subset of X.

Note. The metric space equivalents of Proposition IX.4.14 are Proposition II.4.3(b) for part (a) (see Munkres' Theorem 26.2), and Proposition II.4.3(a) for part (d) (see Munkres' Theorem 26.3); by the way, Conway gives a proof of part (d) on page 225. Parts (b) and (c) don't really have metric space equivalents (we should observe that every metric space is a Hausdorff topological space). Munkres' Lemma 26.1 is related to part (b) and Munkres' Theorem 31.2(a) is part (c).

Note. The following corollary is similar to the separation property called "regular" (or "T_2") which requires that for any closed set Y and any point $a \in X \setminus Y$ there are open sets U and V where $a \in U$, $Y \subset V$, and $U \cap V = \varnothing$. The corollary is similar but requires Y to be compact as opposed to closed (we know every compact set in a Hausdorff space is closed by Proposition IX.4.14(d), but there are closed sets which are not compact).

Corollary IX.4.15. Let $(X, \mathcal{T})$ be a Hausdorff space and let Y be a compact subset of X. Then for each point $a \in X \setminus Y$ there are open subsets U and V of X such that $a \in U$, $Y \subset V$, and $U \cap V = \varnothing$.

Note. Munkres defines a *basis* for a topology on set X as a collection $\mathcal{B}$ of subsets of X such that

(1) For each $x \in X$ there is at least one $B \in \mathcal{B}$ with $x \in B$.

(2) If $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

The *topology $\mathcal{T}$ generated by $\mathcal{B}$* is defined as: A subset U of X is in $\mathcal{T}$ if for each

$x \in U$ there is $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$ (see Munkres, page 78). Conway introduces the same idea, but does so "locally" by first considering a given $x \in X$ and the dealing with basis elements that contain x. He then denotes this "local basis" as $\mathcal{N}_2$ and then defines a *neighborhood system* on X as $\{\mathcal{N}_x \mid x \in X\}$. So Conway's "neighborhood system" on X is the same as Munkres' "basis" for a topology. In both cases, we must confirm that these systems/bases actually generate a topology on X.

Note. To inspire a neighborhood system/basis for a topology, Conway draws a parallel with the generation of a metric topology. First, the metric is used to define open balls and then open ε-balls are used to define open sets. The neighborhood system/basis elements play the role of the open balls and the open sets are then defined in terms of the neighborhood systems/basis elements (in Proposition IX.4.17(b) for Conway).

Definition IX.4.16. Let X be a set and suppose that for each point $x \in X$ there is a collection $\mathcal{N}_x$ of subsets of X having the following properties:

(a) for each $U \in \mathcal{N}_x$ we have $x \in U$;

(b) if $U, V \in \mathcal{N}_x$, then there is a $W \in \mathcal{N}_x$ such that $W \subset U \cap V$;

(c) if $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ then for each $z \in U \cap V$ there is a $W \in \mathcal{N}_x$ such that $W \subset U \cap V$.

The collection $\{\mathcal{N}_x \mid x \in X\}$ is a *neighborhood system* of X.

Note. By letting $x = y = z$ in condition (c) of Definition IX.4.16, we can see that condition (b) follows. So to show that a collection of sets is a neighborhood system, we need only verify parts (a) and (c) of the definition.

Note. Part (b) is the most important part of Proposition IX.4.17 since it shows how to create a topology on X from a neighborhood system on X. This is the same as Munkres' Theorem 13.A (see my notes at: `http://faculty.etsu.edu/gardnerr/5357/notes/Munkres-13.pdf`) for a basis of a topology. We include a proof of Proposition IX.4.17(b).

Proposition IX.4.17.

(a) If $(X, \mathcal{T})$ is a topological space and $\mathcal{N}_x = \{U \in \mathcal{T} \mid x \in U\}$ then $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on X.

(b) If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on a set X then let $\mathcal{T} = \{U \mid x \text{ in } U \text{ implies there is a } V \text{ in } \mathcal{N}_x \text{ such that } V \subset U\}$. Then $\mathcal{T}$ is a topology on X and $\mathcal{N}_x \subset \mathcal{T}$ for each x.

(c) If $(X, \mathcal{T})$ is a topological space and $\{\mathcal{N}_x \mid x \in X\}$ is defined as in part (a), then the topology obtained as in part (b) is again $\mathcal{T}$.

(d) If $\{\mathcal{N}_x \mid x \in X\}$ is a given neighborhood system and $\mathcal{T}$ is the topology defined in part (b), then the neighborhood system obtained from $\mathcal{T}$ contains $\{\mathcal{N}_x \mid x \in X\}$. That is, if V is one of the neighborhoods of x obtained from $\mathcal{T}$ then there is a U in $\mathcal{N}_x$ such that $U \subset V$.

Definition IX.4.18. If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on X and $\mathcal{T}$ is the topology defined in part (b) of Proposition 4.17, then $\mathcal{T}$ is called the *topology induced by the neighborhood system.*

Note. The final result relates properties of neighborhood systems to Hausdorff topologies.

Corollary IX.4.19. If $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system on X and $\mathcal{T}$ is the induced topology then $(X, \mathcal{T})$ is a Hausdorff space if and only if for any two distinct points $x, y \in X$ there is a set $U \in \mathcal{N}_x$ and a set $V \in \mathcal{N}_y$ such that $U \cap V = \varnothing$.

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