IX.5. The Sheaf of Germs of Analytic Functions on an Open Set.

Note. In this section we introduce the Riemann surface of a complete analytic function as a topological space. An analytic structure is put on this topological space in the next section when we consider analytic manifolds.

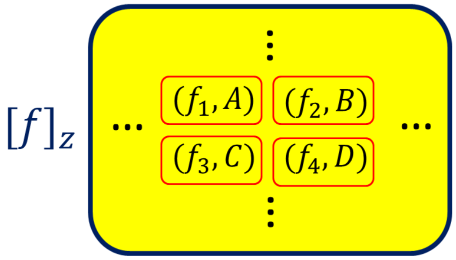
Note. Recall from Section IX.2 that for a function element (f, D) the germ of f at a is the collection of all function elements (g, B) such that $a \in B$ and $f(z) = g(z)$ for all z in the component of $B \cap D$ that contains a, denoted $[f]_a$ (see Definition IX.2.1 and Note IX.2.A).

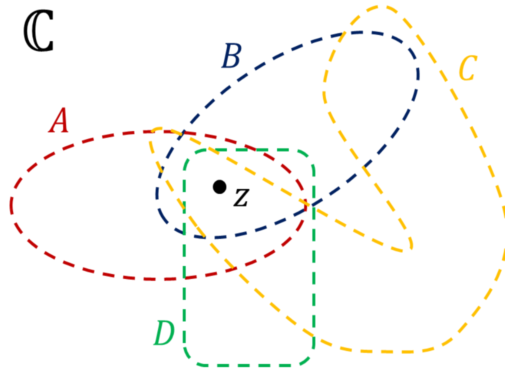
Definition IX.5.1. For an open set $G \subset \mathbb{C}$, let

$$\mathscr{S} = \{(z, [f]_z) \mid z \in G, f \text{ is analytic at } z\}.$$

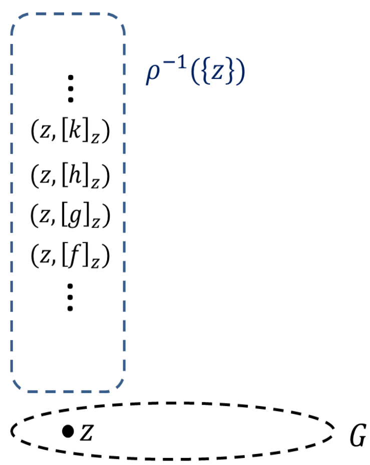
Define $\rho : \mathscr{S}(G) \to \mathbb{C}$ by $\rho(z, [f]_z)) = z$. The pair $(\mathscr{S}(G), \rho)$ is the *sheaf of germs of analytic functions on* G. The map ρ is the *projection map*, and for each $z \in G$, $\rho^{-1}(z) = \rho^{-1}(\{z\})$ is the *stalk* (or *fiber*) over z. Set G is the *base set* of the sheaf.

Note. Based on Conway's "How do we picture this sheaf?" comments on page 228, I propose the following. Each germ of f at $z \in G$, $[f]_z$, consists of function elements:

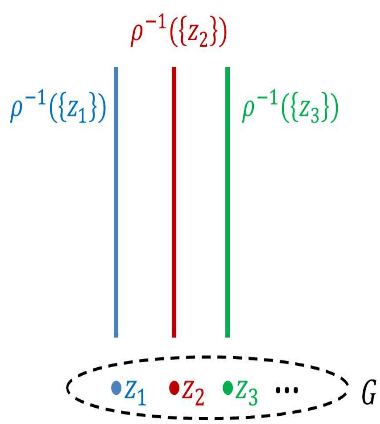




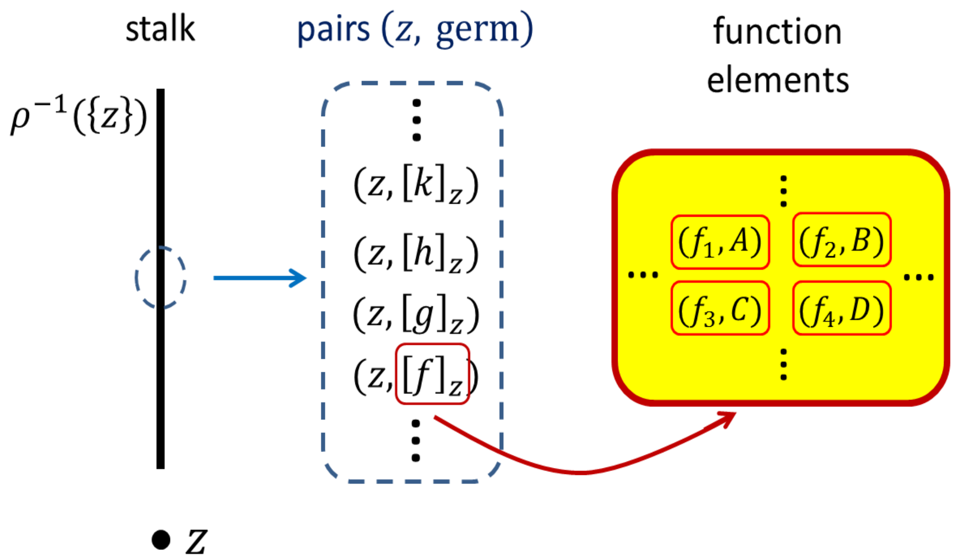
A stalk for a given $z \in G$ consists of ordered pairs $(z, [f]_z)$, so visualize this as a narrow stack of germs lying above point $z \in G$ (though we have no ordering on pairs nor any relationship between the different functions f, g, h, k, except that each is analytic near z):



Finally, as z varies over G we have the sheaf of stalks over G (though the terminology is "sheaf of *germs*"):



So the stalks are make of ordered pairs, the second entry of which is the germ of a function at the point which is the first entry in the ordered pair, and the germ is make up of function elements:



Note. Instead of thinking of the sheaf $\mathscr{S}(G)$ as a bundled collection of stalks, it is more convenient to think of it as a collection of *sheets* lying above G and indexed by the germs of functions (or more simply, as indexed by the functions themselves). So we will let x vary and use analytic continuation to move along the germs in a complete analytic function $\mathscr{F}$ (recall that $[f]_a$ and $[g]_a \in \mathscr{F}$ if there is a path from a to b such that $[g]_b$ is an analytic continuation of $[f]_a$ along the path; see Definition IX.2.7). In this way, a given function element (f, D) is germ $[f]_z$ will lead to a sheet for the function element called the *Riemann surface* of the function (technically, it is the Riemann surface of the complete analytic function $\mathscr{F}$, as we'll see in Definition IX.5.14). We will see that the connected components of sheaf $\mathscr{S}(G)$ are sheets of this kind. So we need a topology on $\mathscr{S}(G)$. We do so by introducing a neighborhood system on G (i.e., a basis for a topology on $\mathscr{S}(G)$).

Definition IX.5.2. Let G be an open set in $\mathbb{C}$ and let $D \subset G$ be open. For $f : D \to \mathbb{C}$ analytic on D, define $N(f, D) = \{(z, [f]_z) \mid z \in D\}$.

Note. The set $N(f, D)$ is the part of the sheet in $\mathscr{S}(G)$ which is indexed by f and which lies above D.

Theorem IX.5.3. For each point $(a, [f]_a)$ in the sheaf $\mathscr{S}(G)$ let

$$\mathcal{N}_{(a,[f]_a)} = \{N(g, B) \mid a \in B \text{ and } [g]_a = [f]_a\}.$$

Then $\{\mathcal{N}_{(a,[f]_a)} \mid (a, [f]_a) \in \mathscr{S}(G)\}$ is a neighborhood system on $\mathscr{S}(G)$ and the induced topology is Hausdorff. Furthermore, the induced topology makes the projection map $\rho : \mathscr{S}(G) \to G$ continuous.

Note. In the proof of Theorem IX.5.3, where we showed that the induced topology is Hausdorff, we had two cases for $(a, [f]_a) \neq (b, [g]_b)$ for elements $(a, [f]_a), (b, [g]_n) \in \mathscr{S}(G)$. The first case is $a \neq b$ in which case the points are on different stalks, namely $\rho^{-1}(a)$ and $\rho^{-1}(b)$ respectively. The second case is $a = b$ that $[f]_1 \neq [g]_1$. In this case, the points are on the same stalk but were separated by $N(f, D)$ and $N(g, D)$ so that they are at different places on stalk $\rho^{-1}(a)$.

Note. The next definition and three results involve some additional topological concepts that will be applied to the topology on $\mathscr{S}(G)$.

Definition IX.5.5. Let $(X, \mathcal{T})$ be a topological space. If $x_0, x_1 \in X$ then an *arc* (or *path*) in X from x_0 to x_1 is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The point x_0 is the *initial point* of γ and x_1 is the *final point* (or *terminal point*). The *trace* of γ is the set $\{\gamma\} = \{\gamma(t) \mid 0 \le t \le 1\}$. A subset A of X is *arcwise connects* (or *pathwise connected*) if for any two points $x_0, x_1 \in A$ there is a path from x_0 to x_1 whose trace lies in A. The topological space $(X, \mathcal{T})$ is *locally arcwise connected* (or *locally pathwise connected*) if for each $x \in X$ and each open set $U \in \mathcal{T}$ where $x \in U$, there is an open arcwise (pathwise) connected set V such that $x \in V$ and $V \subset U$.

Lemma IX.5.A. Let $(X, \mathcal{T})$ be a topological space. For each $x \in X$ let $\mathcal{X}_x$ be the collection of all open arcwise connected (pathwise connected) subsets of X which contain x. Then X is locally arcwise connected if and only if $\{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system which induces topology $\mathcal{T}$ on X.

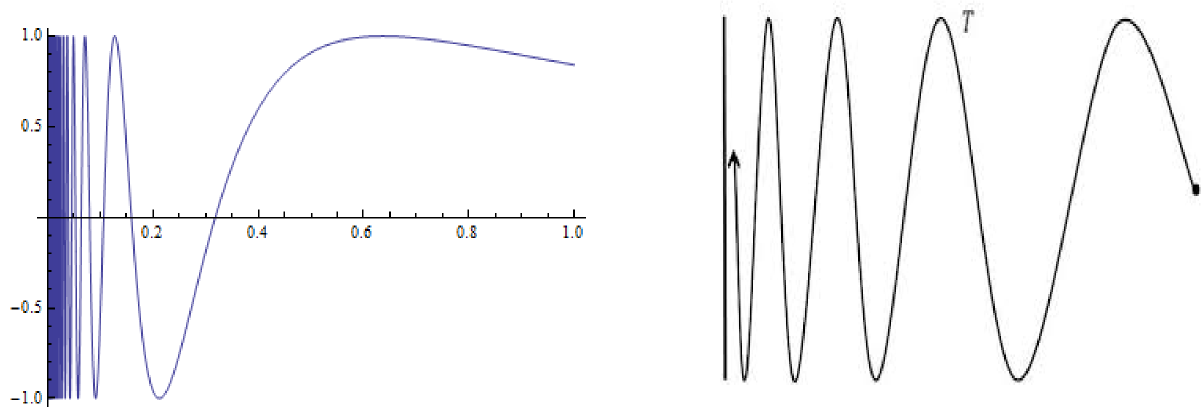
Note. The proofs of Lemma IX.5.A and the next result are left as Exercises IX.5.A and IX.5.B. Munkres in *Topology*, 2nd Edition (Prentice Hall, 2000) addresses path connectedness and local path connectedness in his Sections 24 ("Connected Subspaces of the Real Line") and 25 ("Components and Local Connectedness"). Proposition IX.5.6 is related to Munkres' Theorems 25.4 and 25.5.

Proposition IX.5.6. Let $(X, \mathcal{T})$ be a topological space.

(a) If A is an arcwise connected subset of X then A is connected.

(b) If X is locally arcwise connected then each component of X is an open set.

Note. The converse of Theorem IX.5.6(a) does not hold. Consider $X = \{t + i\sin(1/t) \mid t > 0\} \cup \{si \mid -1 \leq s \leq 1\}$:



A to-scale graph of the topologist's sine curve and a schematic. From `http://hyperspacewiki.org/images/1/15/Topologistsinecurve.png` and `http://faculty.mccneb.edu/akriesel/AmandaKrieselAPP3.pdf`, respectively.

Now $X_1 = \{t + i\sin(1/t) \mid t > 0\}$ is an arcwise connected set and so by Theorem IX.5.6(a) is a connected set. Set X is the closure of X_1 and so by Proposition II.2.8, X is connected. But X is not arcwise connected since there is no arc in X from any point in X_1 to any point in $X_2 \setminus \{0\} = \{si \mid -1 \leq s \leq 1\} \setminus \{0\}$.

Note. The proof of the following is "virtually identical" to the proof of Theorem II.2.3 (which concerns the "polygonal connectivity;; of open sets in $\mathbb{C}$).

Proposition IX.5.7. If X is locally arcwise connected then an open connected subset of X is arcwise connected.

Note. We now get back to topological results concerning the sheaf of germs of analytic functions on G, $\mathscr{S}(G)$. We use arcwise connectivity to introduce analytic continuation.

Proposition IX.5.8. Let G be an open subset of the complex plane and let U be an open connected subset of G such that there is an analytic function f defined on U. Then $N(f,U) = \{(z,[f]_z) \mid z \in U\}$ is arcwise connected in $\mathscr{S}(G)$.

Corollary IX.5.9. The sheaf of germs of analytic functions on G, $\mathscr{S}(G)$, is locally arcwise connected and the components of $\mathscr{S}(G)$ are open arcwise connected sets.

Note. The next result is big since it relates connected components of $\mathscr{S}$ to analytic continuation (and conversely).

Theorem IX.5.10. There is a path in $\mathscr{S}(G)$ from $(a,[f]_a)$ to $(b,[g]_b)$ if and only if there is a path γ in G from a to b such that $[g]_b$ is an analytic continuation of $[f]_a$ along γ.

Theorem IX.5.11. Let $\mathscr{C} \subset \mathscr{S}(G)$ and let $(a,[f]_a) \in \mathscr{C}$. Then $\mathscr{C}$ is a component of $\mathscr{S}(G)$ if and only if

$$\mathscr{C} = \{(b,[g]_b) \mid [g]_b \text{ is the continuation of } [f]_a \text{ along some curve in } G\}.$$

Note. We are now ready to define a Riemann surface! First, we start with a function element (f, D) where D is a region in $\mathbb{C}$ and f is analytic on D. Next we consider the complete analytic function obtained from the element (f, D),

$$\mathscr{F} = \{[g]_b \mid \text{ for some } a \in D \text{ there is a path } \gamma \text{ from } a \text{ to } b \text{ and}$$

$$[g]_b \text{ is an analytic continuation of } [f]_a \text{ along } \gamma\}.$$

Now we define $G \subset \mathbb{C}$ as the set of all z values at which there is an analytic continuation of f:

$$G = \{z \mid \text{ there is a germ } [g]_z \in \mathscr{F}\}.$$

Notice that G is open since for any $z \in G$ there is a germ $[g]_z \in \mathscr{F}$ and g is defined on an open disk B containing x and so $z \in B \subset G$. Since $\mathscr{F}$ consists of all analytic continuations of $[f]_a$, we have by Proposition IX.5.11 that

$$\mathscr{R} = \{(z, [g]_z) \mid [g]_z \in \mathscr{F}\}$$

is a component of $\mathscr{S}(G)$ where we have $\rho(\mathscr{R}) = G$. We might want to think of $\mathscr{R}$ as the Riemann surface of $[f]_a$, but it is slightly more complicated.

Definition IX.5.14. Let $\mathscr{F}$ be a complete analytic function. With $\mathscr{R} = \{(z, [g]_z) \mid [g]_z \in \mathscr{F}\}$ and $\rho : \mathscr{S}(\mathbb{C}) \to \mathbb{C}$ defined by $\rho(z, [f])z)) = z$ as the projection map of the sheaf of germs into $\mathbb{C}$, then the pair $(\mathscr{R}, \rho)$ is the *Riemann surface* of $\mathscr{F}$. The open set $G = \{z \mid \text{ there is a germ } [g]_z \text{ in } \mathscr{F}\}$ is the *base space* of $\mathscr{F}$.

Note. In Definition IX.5.14, we consider initially the sheaf of germs $\mathscr{S}(\mathbb{C})$. However, once set G is defined this can be replaced with sheaf $\mathscr{S}(G)$.

Theorem IX.5.15. Let $\mathscr{F}$ be a complete analytic function with base space G and let $(\mathscr{R}, \rho)$ be its Riemann surface. Then $\rho : \mathscr{R} \to G$ is an open continuous map. Also, if $(a, [f]_a)$ is a point in $\mathscr{R}$ then there is a neighborhood $N(f, D)$ of $(a, [f]_a)$ such that ρ maps $N(f, D)$ homeomorphically onto an open disk in $\mathbb{C}$.

Note. If you have seen Riemann surfaces before, you may have considered analytic function mapping Riemann surfaces into $\mathbb{C}$. In the next section we will see this kind of behavior, but it will be the complete analytic function $\mathscr{F}$ that maps $\mathscr{R}$ to $\mathbb{C}$.

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