IX.6. Analytic Manifolds.

Note. In this section we define a topological space (an "analytic manifold") on which we define a complete analytic function from the manifold into $\mathbb{C}$. In this sense, we will map a Riemann surface into $\mathbb{C}$ in an analytic way (see Theorem IX.6.9). We extend the Maximum Modulus Theorem, Liouville's Theorem, and the Open Mapping Theorem to this setting.

Note. We start with a motivational example. Consider the extended plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. If $a \in \mathbb{C}_\infty$ and $a \neq \infty$ then a *finite* neighborhood U_a of a is an open subset of the plane $\mathbb{C}$. We define $\varphi_a : U_a \to \mathbb{C}$ as the identity map $\varphi_a(z) = z$. Then φ_a gives a "coordinatization" of the neighborhood U_a (though trivially). If $a = \infty$ then let $U_\infty = \{z \mid |z| > 1\} \cup \{\infty\}$ and define $\varphi_\infty : U_\infty \to \mathbb{C}$ as

$$\varphi_\infty(z) = \begin{cases} 1/z & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \end{cases}.$$

Notice φ_∞ is a homeomorphism (see Definition VIII.2.1) of U_∞ onto $B(0;1)$. So to each point $a \in \mathbb{C}_\infty$ there is a pair (U_a, φ_a) such that U_a is a neighborhood of a and φ_a is a homeomorphism of U_a onto an open subset of the plane. Notice that since each φ_a is one to one then φ_a^{-1} exists (we have $\varphi_a : \mathbb{C}_\infty \to \mathbb{C}$ and $\varphi_a^{-1} : \mathbb{C} \to \mathbb{C}_\infty$). We now consider what happens when $U_a \cap U_b \neq \varnothing$. First, suppose $a \neq \infty$ and $U_a \cap U_\infty \neq \varnothing$. Let $G_\infty = B(0;1) = \varphi_\infty(U_\infty)$ and let $G_a = \varphi_a(U_a) = U_a$. Then

$$\varphi_\infty^{-1}(z) = \begin{cases} 1/z & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \end{cases}$$

for all $z \in G_\infty$, and so $\varphi_a \circ \varphi_\infty^{-1}(z) = 1/z$ for all $z \in \varphi_\infty(U_\infty \cap U_a)$ (since $\infty \notin U_a$ then $0 \notin \varphi_\infty(U_\infty \cap U_a) \subset \mathbb{C}$ and $\varphi_a \circ \varphi_\infty^{-1}(z) = 1/z$ is analytic on $\varphi_\infty(U_\infty \cap U_a)$). Similarly $\varphi_\infty \circ \varphi_a^{-1}(z) = 1/z$ for all $z \in \varphi_a(U_a \cap U_\infty)$ and $\varphi_\infty \circ \varphi_a^{-1}(z) = 1/z$ is analytic on $\varphi_a(U_a \cap U_\infty)$. If $a = \infty$ then $U_a \cap U_\infty = U_\infty$ and $\varphi_a \circ \varphi_\infty^{-1}(z) = \varphi_\infty \circ \varphi_a^{-1}(z) = z$ is analytic on $\varphi_\infty(U_\infty \cap U_a) = \varphi_a(U_a \cap U_\infty) = G_\infty = B(0;1)$. If both a and b are finite then $\varphi_a \circ \varphi_b^{-1}(z) = z$ for all $z \in \varphi_b(U_b \cap U_a) = U_b \cap U_a$. In each case, notice that $\varphi_a \circ \varphi_b^{-1} : \mathbb{C} \to \mathbb{C}$ is analytic on its domain $\varphi_b(U_b \cap U_a) = \varphi_b(U_a \cap U_b)$. Also, each $\varphi_a : \mathbb{C}_\infty \to \mathbb{C}$. This example inspires the following two definitions.

Definition IX.6.1. Let X be a topological space. A *coordinate patch* on X is a pair (U, φ) where U is an open subset of X and φ is a homeomorphism of U onto an open subset of $\mathbb{C}$. If $a \in U$ then the coordinate patch $(U, \varphi) = (U_a, \varphi_a)$ is said to *contain* a.

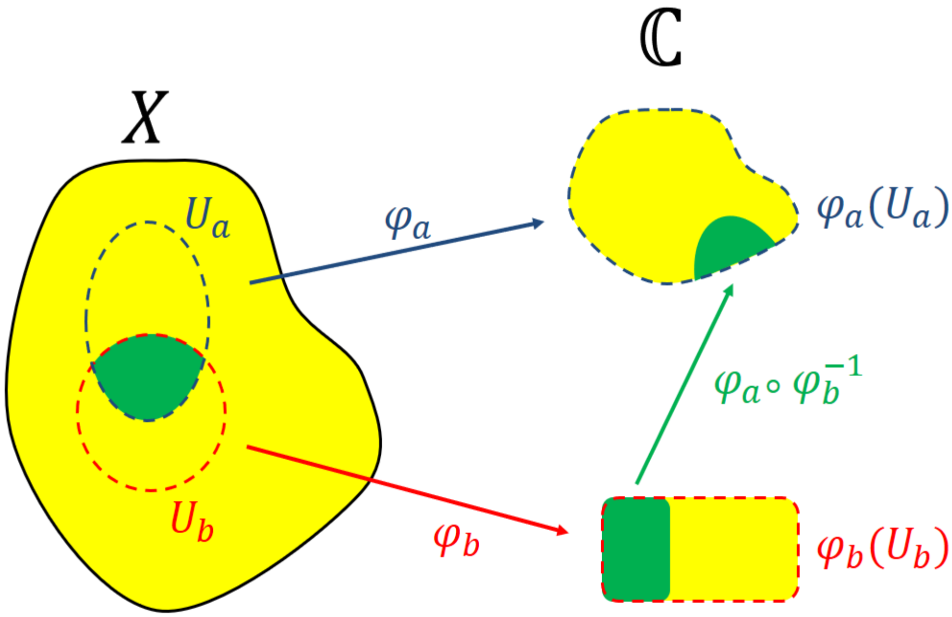
Definition IX.6.2. An *analytic manifold* (or *analytic surface*) is a pair (X, Φ) where X is a Hausdorff connected topological space and Φ is a collection of coordinate patches on X such that

(i) each point of X is contained in at least one member of Φ, and

(ii) if $(U_a, \varphi_a), (U_b, \varphi_b) \in \Phi$ with $U_a \cap U_b \neq \varnothing$ then $\varphi_a \circ \varphi_b^{-1}$ is an analytic function of $\varphi_b(U_a \cap U_b) \subset \mathbb{C}$ onto $\varphi_a(U_a \cap U_b) \subset \mathbb{C}$.

The set Φ of coordinate patches is an *analytic structure* on X.

Note. The relationships between the functions and sets are as follows:



Note. We follow Conway's terminology and use the term "analytic surface" for the structures studied here. Since $\varphi_1 \circ \varphi_b^{-1}$ maps subsets of $\mathbb{C}$ into $\mathbb{C}$ (analytically), then the term "surface" or "1-dimensional complex manifold" is appropriate. In the event we considered analytic functions mapping subsets of $\mathbb{C}^n$ into $\mathbb{C}^n$, then we could consider n dimensional complex manifolds.

Note. Trivially, $\mathbb{C}$ is an analytic surface. We see that $\mathbb{C}_\infty$ is also an example of an analytic surface.

Note. By convention, we impose one more condition on the collection of coordinate patches Φ. First, we need a preliminary result.

Proposition IX.6.3. Let (X, Φ) be an analytic surface.

(a) Let V be an open connected subset of X. If

$$\Phi_V = \{(U \cap V, \varphi) \mid (U, \varphi) \in \Phi\}$$

then (V, Φ_V) is an analytic surface.

(b) If Ω is a topological space such that there is a homeomorphism h of X onto Ω then with

$$\Phi = \{(h(U), \varphi \circ h^{-1}) \mid (U, \varphi) \in \Phi\}$$

we have that (Ω, Ψ) is an analytic surface.

Note. Consider the 2-sphere $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ of Section 1.6 under the subspace topology where $S \subset \mathbb{R}^3$ and $\mathbb{R}^3$ has the usual metric topology. S is homeomorphic to $\mathbb{C}_\infty$, so by Proposition IX.6.3(b), S is an analytic surface since $\mathbb{C}_\infty$ is.

Note. By Proposition IX.6.3(a), if $(U, \varphi) \in \Phi$ and V is an open subset off U then (V, φ) is a coordinate patch on X (technically, we are using φ restricted to V, $\varphi|_V$, here; Conway states that "...for the sake of brevity, care will not be taken in mentioning the appropriate domain of a function ..." [page 234]). By convention, we take the following as a third property of analytic surfaces:

Definition IX.6.2 (continued). An *analytic manifold* (or *analytic surface*) is a pair (X, Φ) where X is a Hausdorff connected topological space and Φ is a collection of coordinate patches on X such that (i) and (ii) above hold and:

(iii) if $(U, \varphi) \in \Phi$ and V is an open subset of U then $(V, \varphi) \in \Phi$.

Note. When giving an analytic structure Φ on a Hausdorff connected topological space X, we only need to give the coordinate patches that "generate" Φ in the sense given in Definition IX.6.3(iii).

Note. Other sources deal with Definition IX.6.2(iii) in different ways. Similar to Conway's approach, Wald in his *General Relativity* takes it by convention that Φ is maximal in the sense of condition (iii) (in order to avoid the complication of defining new manifolds from given manifolds by simply adding or deleting a coordinate system). See my online notes:

`http://faculty.etsu.edu/gardnerr/5310/notes-Wald/waldrel-2-1.pdf`.

For example, Hawking and Ellis in *The Large Scale Structure of Space-Time* require that "atlas" Φ be a "complete atlas"; that is, it should satisfy condition (iii). See my online notes:

`http://faculty.etsu.edu/gardnerr/5310/Notes-Hawking-Ellis/`
`Hawking-Ellis-2-1.pdf`.

Note. Conway gives a definition of a real differentiable (or "C^1") surface in $\mathbb{R}^3$.

Definition. Let $X \subset \mathbb{R}^3$. X is a *differentiable real 2-manifold* if each point in X is contained in a coordinate patch (U, φ), where $\varphi : U \to \mathbb{R}^2$, such that $\varphi^{-1} : \varphi(U) \to U \subset \mathbb{R}^3$ has coordinate functions with continuous partial derivatives (more appropriately, X is a "C^1" real 2-manifold in $\mathbb{R}^3$). That is, with $G = \varphi(U)$ and $\varphi^{-1} : \mathbb{R}^2 \to U$ satisfies $(s,t) \in G$ and ξ, η, ζ have continuous first partial derivatives.

Note. Conway observes (page 235) that: "A folded piece of paper is not a differentiable real 2-manifold. In fact, if (U, φ) is a patch that contains a point on the crease then φ^{-1} has at least one non-differentiable coordinate function." Yet a "folded piece of paper" (provided it is an open set) is homeomorphic to a region in $\mathbb{R}^2$ and so, by Proposition IX.6.3(b) is an analytic surface. The reason we have analytic on the one (complex) hand and don't even have differentiability on the other (real) hand is because the homeomorphism h of Proposition IX.6.3(b) imposes the analytic structure on the "piece of paper." But in the real setting, there is already a "differentiable structure" in $\mathbb{R}^3$ and this structure yields the nondifferentiability at the crease. We can similarly show using Proposition IX.6.3(b) that a cube in $\mathbb{R}^3$ is an analytic surface since it is homeomorphic to the 2-sphere $S \subset \mathbb{R}^3$ which, as observed above, is homeomorphic to $\mathbb{C}_\infty$ which is an analytic surface.

Definition. Let G be a region in $\mathbb{C}$ and let $f : G \to \mathbb{C}$ be an analytic function. The *graph* of f on G is $\Gamma = \{(z, f(z)) \mid z \in G\} \subset \mathbb{C}^2$.

Note. We wish to put an analytic structure on the graph Γ. We will require $f'(z) \neq 0$ for all $z \in G$ so that, by Corollary IV.7.6, f is one to one.

Note. The projection $p(z, f(z)) = z$ maps $\Gamma \to G$. Now a basis for the topology on $\mathbb{C}$ is $\{B(z; r) \mid z \in \mathbb{C}, r > 0\}$. So a basis for $\mathbb{C} \times \mathbb{C}$ is $\{B(z_1; r_z) \times B(z_2; r_2) \mid z_1, z_2 \in \mathbb{C}, r_1 > 0, r_2 > 0\}$ (see my online Topology notes for Section 15, "The Product Topology on $X \times Y$," `http://faculty.etsu.edu/gardnerr/5357/notes/Munkres`

`-15.pdf`). So for $\gamma \subset \Gamma$ an open set, $\gamma = \gamma \cap \left(\cup_{j\in J} B(z_j; r_j) \times B(z_j'; r_j')\right)$ for some indexing set J (under the subspace topology on G). Then $p(\gamma) = p(\Gamma) \cap (\cup_{j\in J} B(z_j; r_j))$ is open in G (and so is open in $\mathbb{C}$, since G is open in $\mathbb{C}$). If $g \subset G$ is open then (since G is open in $\mathbb{C}$ then g is open in $\mathbb{C}$), $g = \cup_{j\in J} B(z_j; r_j)$ for some indexing set J. Since f is analytic, then by the Open Mapping Theorem (Theorem IV.7.5), $f(g)$ is open in $\mathbb{C}$. Now $p^{-1}(g) = \{(z, f(z)) \mid z \in g\} = \Gamma \cap (g \times f(g))$. Since $g \times f(g)$ is open in $\mathbb{C}^2$ then $p^{-1}(g)$ is open in the subspace topology on Γ So $p : \Gamma \to G$ is one to one, onto, continuous, and p^{-1} is continuous. That is, p is a homeomorphism of Γ with G and p^{-1} is a homeomorphism of G with Γ. So, by Proposition IX.6.3(b), Γ is an analytic surface. But the analytic structure on Γ is based on the analytic structure of G given in Proposition IX.6.3(b) which is only based on the analytic structure of open set $G \subset \mathbb{C}$ and the projection homeomorphism p. We want an analytic structure on Γ that is somehow based on the analytic function f. This is basic in our exploration of Riemann surfaces and is addressed in the next three results.

Proposition IX.6.6. Let G be a region in the plane and let f be an analytic function on g with non-vanishing derivative. For $\alpha = (a, f(a)) \in \Gamma = \{(z, f(z)) \mid z \in G\}$. Let D_z be a disk about a such that $D_a \subset G$ and f is one to one on D_a (which is possible since $f'(a) \neq 0$). Let $U_\alpha = \{(z, f(z)) \mid z \in D_a\}$ and define $\varphi_\alpha : U_\alpha \to \mathbb{C}$ by $\varphi_\alpha(z, f(z)) = f(z)$ for each $(z, f(z)) \in U_\alpha$. If Γ is the graph of f and $\Phi = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in \Gamma\}$ then (Γ, Φ) is an analytic surface.