Supplement: Laurent Series Development (Theorem V.1.11)

Note. In this section, we give a detailed proof of Theorem V.1.11, the "Laurent Series Development" theorem. First, we recall some previous results which are needed in the proof of this theorem.

Theorem III.1.3. If $\sum_{n=0}^{\infty} a_n(z-a)^n$, define the number R as $\frac{1}{R} = \overline{\lim}\, |a_n|^{1/n}$ (so $0 \le R \le \infty$). Then

(a) if $|z-a| < R$, the series converges absolutely,

(b) if $|z-a| > R$, the series diverges, and

(c) if $0 < r < R$ then the series converges uniformly on $|z-a| \le r$. Moreover, R is the only number having properties (a) and (b). R is called the *radius of convergence* of the power series.

Theorem IV.2.8. Let f be analytic in $B(a;R)$. Then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z-a| < R$ where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence $\ge R$.

Lemma IV.5.1. Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \ge 1$ let $F_m(z) = \int_\gamma \varphi(w)(w-z)^{-m}\,dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

Corollary IV.5.9. (Theorem 5.8 with one curve.) Let G be an open set and $f : G \to \mathbb{C}$ analytic. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$ then for $a \in G \setminus \{\gamma\}$

$$f^{(k)}(a)n(\gamma; a) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{k+1}}\, dz.$$

Theorem V.1.2. If f has an isolated singularity at a then $z = a$ is a removable singularity if and only if $\lim_{z \to a}(z-a)f(z) = 0$.

Note. Now for our main result.

Theorem V.1.11. Laurent Series Development.

Let f be analytic in $\text{ann}(a; R_1, R_2)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

where the convergence is absolute and uniform over the closure of $\text{ann}(a; r_1, r_2)$ if $R_1 < r_1 < r_2 < R_2$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw \qquad (1.12)$$

where γ is the circle $\gamma(t) = a + re^{it}$ where $t \in [0, 2\pi]$ for any r with $R_1 < r < R_2$. Moreover, this series is unique.

Proof. If $R_1 < r_1 < r_2 < R_2$ and γ_1, γ_2 are the circles $\gamma_1(t) = a + r_1 e^{it}$ for $t \in [0, 2\pi]$ and $\gamma_2(t) = a + r_2 e^{it}$ for $t \in [0, 2\pi]$, then $\gamma_1 \sim \gamma_2$ in $\text{ann}(a; R_1, R_2)$. Then by Cauchy's Theorem (Third Version, Theorem IV.6.7) for any analytic g on ann$(a; R_1, R_2)$, we have $\int_{\gamma_1} g = \int_{\gamma_2} g$. So the integral in (1.12) is independent of r, so for each $n \in \mathbb{Z}$, a_n is a constant (i.e., independent of γ).

Claim. *Definition of f_1 and f_2; $f(z) = f_1(z) + f_2(z)$.*

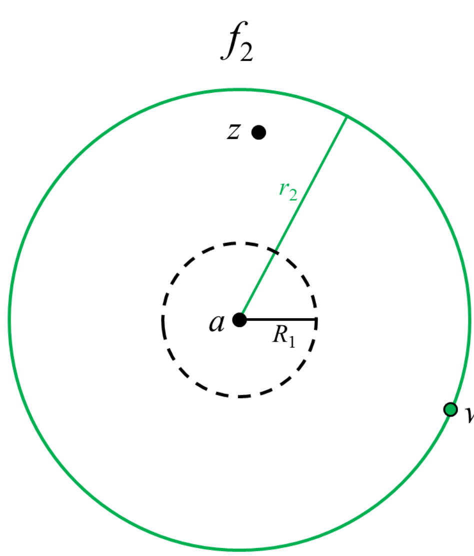
Moreover, function $f_2 : B(a; R_2) \to \mathbb{C}$ given as

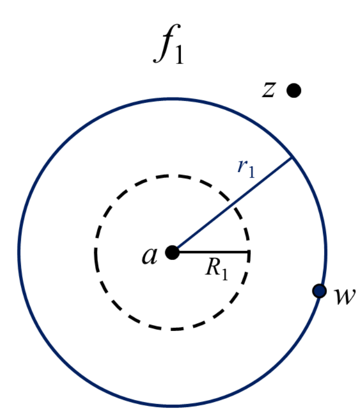
$$f_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z}\, dw,$$

where $|z - a| < r_2$ and $R_1 < r_2 < R_2$, is a well defined function (independent of r_2). By Lemma IV.5.1 (with $m = 1$ and $\varphi(w) = f(w)$), f_2 is analytic in $B(a; r_2)$. Similarly, if $G = \{z \mid |z - a| > R_1\}$ then $f_1 : G \to \mathbb{C}$ defined as

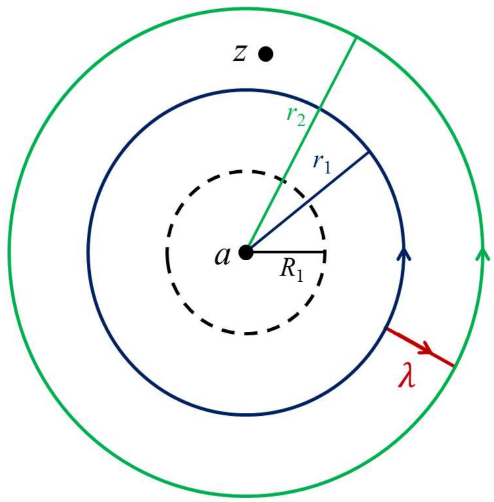
$$f_1(z) = \frac{-1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw,$$

where $|z-a| > r_1$ (so r_1 is chosen such that $z \notin \{\gamma_1\}$) and $R_1 < r_1 < R_2$ is analytic in G.





If $R_1 < |z - a| < R_2$, fix r_1 and r_2 so that $R_1 < r_1 < |z - a| < r_2 < R_2$. Let $\gamma_1(t) = a + r_1 e^{it}$ for $t \in [0, 2\pi]$ and $\gamma_2(t) = a + r_2 e^{it}$ for $t \in [0, 2\pi]$, as above. Let λ be a line segment from a point on γ_1 radially to a point on γ_2 which does not contain z:



Since $\gamma_1 \sim \gamma_2$ on $\text{ann}(a; R_1, R_2)$, we have that the closed curve $\gamma = \gamma_2 - \lambda - \gamma_1 + \lambda$ is homotopic to 0. Next, $n(\gamma_2; z) = 1$ and $n(\gamma_1; z) = 0$ gives:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \text{ by the Cauchy's Integral Formula} \\ &= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw \text{ since } \gamma \text{ is piecewise smooth} \\ &= f_2(z) + f_1(z) \text{ by the definitions of } f_1 \text{ and } f_2. \end{aligned}$$

We now create series for f_1 (with negative powers of $(z-a)$) and f_2 (with the usual positive powers of $(z-a)$). □

Claim. *The a_n are as claimed for $n \geq 0$.*

Since f_2 is analytic in $B(a; R_2)$ then by Theorem IV.2.8

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ and } a_n = \frac{f^{(n)}(a)}{n!}.$$

Since $f_2(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw$, in Lemma IV.5.1 with $m = 1$ we have $F_1(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{\varphi(w)}{w-z}\,dw$ where $\varphi(w) = f(w)$, and therefore by induction

$$f_2^{(n)}(z) = n! F_{n+1}(z) = \frac{n!}{2\pi i}\int_{\gamma_2} \frac{\varphi(w)}{(w-z)^{n+1}}\,dw.$$

We then have

$$a_n = \frac{f^{(n)}(a)}{n!} = F_{n+1}(a) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}}\,dw,$$

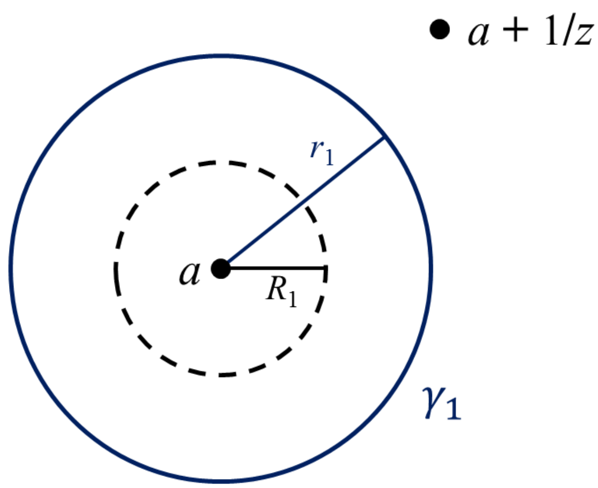
as claimed. $\square$

Claim. *Definition of $g(z)$; $g(z)$ is analytic for $z \in B(0;1/R_1)$.*

Now define $g(z)$ for $0 < |z| < 1/r_1$ as $g(z) = f_1(a + 1/z)$. Notice that $a + 1/z = a - (-1/z)$ and $|-1/z| > r_1$, so $a + 1/z \in \text{ann}(a; r_1, \infty)$. Since f_1 is defined as

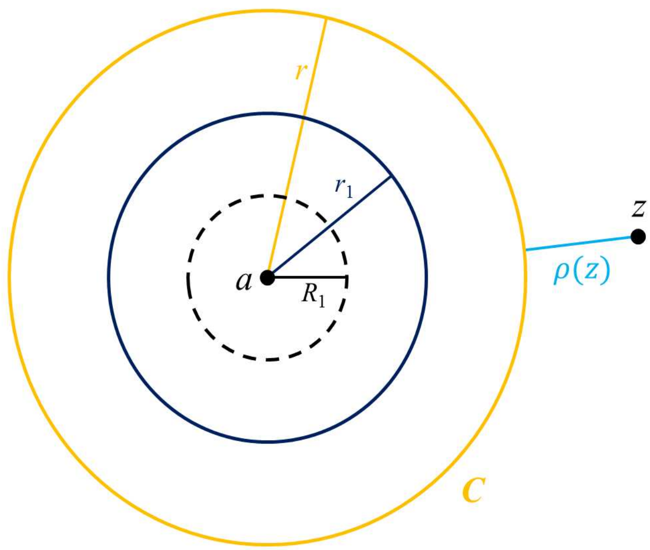
$$f_1(z) = \frac{-1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw$$

where $\gamma_1(t) = a + r_1 e^{it}$ with $t \in [0, 2\pi]$, then $f_1(a + 1/z)$ is defined as long as $a + 1/z \notin \{\gamma_1\}$. We have:



Since $f_1(z)$ is analytic for all z with $|z - a| > r_1$ (from the definition of f_1), then $g(z)$ is analytic for all $a + 1/z$ with $|(a + 1/z) - a| > r_1$, or $|1/z| > r_1$ or $0 < |z| < 1/r_1$. So g has an isolated singularity at $z = 0$. We now show g has a removable singularity at $z = 0$ by showing $\lim_{z\to 0} g(z)$ exists (and hence $\lim_{z\to 0} zg(z) = 0$, as required by Theorem V.1.2 for g to have a removable singularity at $z = 0$). Suppose $R_2 > r > r_1$, $z \in \text{ann}(a; r, \infty)$, and $\rho(z) = d(z, C)$ where C is the circle

$\{w \mid |w-a| = r\}$:



Define $M = \max\{|f(w)| \mid w \in C\}$ and give C a counterclockwise orientation. Then for such z "outside" of C,

$$\begin{aligned} |f_1(z)| &= \left| \frac{-1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z}\, dw \right| \\ &= \left| \frac{-1}{2\pi i} \int_{C} \frac{f(w)}{w-z}\, dw \right| \text{ since } \gamma_1 \sim C \text{ on } \operatorname{ann}(a; R_1, R_2) \\ &\leq \frac{1}{2\pi} \int_C \frac{|f(w)|}{|w-z|}\, |dw| \leq \frac{1}{2\pi} \frac{M 2\pi r}{\rho(z)} = \frac{Mr}{\rho(z)}. \end{aligned}$$

Now $\lim_{z\to\infty} \rho(z) = \infty$ and M, r are constants, so $\lim_{z\to\infty} f_1(z) = 0$. Therefore,

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} f_1(a + 1/z) = \lim_{z\to\infty} f_1(z) = 0.$$

Since $\lim_{z\to 0} g(z) = 0$, g has a removable singularity at $z = 0$ by Theorem V.1.2. We define $g(0) = 0$ and then g is analytic on $B(0; 1/r_1)$ (by the definition of removable singularity). Notice that $r_1 > R_1$ was arbitrary above, and we can conclude that each claim is valid for r_1 replace with R_1 (as in the text). So g can be written $g(z) = \sum_{n=1}^{\infty} B_n z^n$ ($B_0 = 0$ since $g(0) = 0$). So, $f_1(a + 1/z) = g(z) = \sum_{n=1}^{\infty} B_n a^n$

for $z \in B(0;1/R_1)$. $\square$

Claim. *The a_n are as claimed for $n \le -1$.*

We have by definition that $f_1(z) = \frac{-1}{2\pi i}\int_{\gamma_1} \frac{f(x)}{w-z}\,dw$ for $|z-a| > R_1$ where $\gamma_1(t) = a + r_1 e^{it}$ for $t \in [0, 2\pi]$. So for $|a| < R_1$ we have

$$f_1\left(a+\frac{1}{z}\right) = \frac{-1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-\left(a+\frac{1}{z}\right)}\,dw.$$

Replacing w with $a+1/w$, dw with $-w^{-2}\,dw$, and using the inverse of this mapping to take γ_1 to γ_1' (so that the integrand receives the same values of w) we have

$$\gamma_1'(t) = \frac{1}{\gamma_1(t)-a} = \frac{1}{(a+r_1e^{it})-a} = \frac{1}{r_1}e^{-it} \text{ for } t \in [0, 2\pi].$$

Then $f_1(a+1/z)$ becomes

$$\begin{aligned} f_1\left(a+\frac{1}{z}\right) &= \frac{-1}{2\pi i}\int_{\gamma_1'} \frac{f(a+1/w)}{\left(a+\frac{1}{w}\right)-\left(a+\frac{1}{z}\right)}[-w^{-2}]\,dw = \frac{1}{2\pi i}\int_{\gamma_1'} \frac{f(a+1/w)}{w\left(1-\frac{w}{z}\right)}\,dw \\ &= \frac{1}{2\pi i}\int_{\gamma_1'} \frac{zf(a+1/w)}{w(z-w)}\,dw = \frac{1}{2\pi i}\int_{-\gamma_1'} \frac{zf(a+1/w)}{w(w-z)}\,dw \\ &= \frac{z}{2\pi i}\int_{-\gamma_1'} \frac{f(a+1/w)/w}{w-z}\,dw = g(z) \text{ for } 0 < |z| < 1/R_1. \end{aligned}$$

As established above, $g(z)$ is analytic on $B(0;1/R_1)$, so $g(z) = \sum_{n=1}^{\infty} B_n z^n$ (notice that $B_0 = 0$ since $g(0) = 0$). Also, by Theorem IV.2.8, $B_n = g^{(n)}(0)/n!$. We now calculate $g^{(n)}(0)$ using Lemma IV.5.1 and the representation of $g(z)$ as

$$g(z) = \frac{z}{2\pi i}\int_{-\gamma_1'} \frac{f(a+1/w)/w}{w-z}\,dw$$

(notice that $-\gamma_1'(t) = \frac{1}{r_1}e^{it}$ for $t \in [0, 2\pi]$). From the Product Rule, Lemma IV.5.1 (with $\varphi(w) = f(a+1/w)/w$), and Math Induction we can show that

$$g^{(n)}(z) = \frac{n!}{2\pi i}\int_{-\gamma_1'} \frac{f(a+1/w)/w}{(w-z)^n}\,dw + \frac{n!z}{2\pi i}\int_{-\gamma_1'} \frac{f(a+1/w)/w}{(w-z)^{n+1}}\,dw \text{ for } |z| < R_1.$$

So $g^{(n)}(0) = \frac{n!}{2\pi i}\int_{-\gamma_1'} \frac{f(a+1/w)}{w^{n+1}}\,dw$, so $B_n = \frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{-\gamma_1'} \frac{f(a+1/w)}{w^{n+1}}\,dw$. Replacing w with $1/(w-a)$ and dw with $-(w-a)^{-2}\,dw$ and using the inverse of this mapping to take γ_1' to γ_1 (similar to above)

$$\begin{aligned} B_n &= \frac{1}{2\pi i}\int_{-\gamma_1'} \frac{f(a+1/w)}{w^{n+1}}\,dw = \frac{1}{2\pi i}\int_{-\gamma_1} \frac{f(w)}{\left(\frac{1}{w-a}\right)^{n+1}}\left[\frac{-1}{(w-a)^2}\right]dw \\ &= \frac{-1}{2\pi i}\int_{-\gamma_1} f(w)(w-a)^{n-1}\,dw = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(w-a)^{-n+1}}\,dw. \end{aligned}$$

Defining a_{-n} as B_n (and replacing n with $-n$ in B_n) we have

$$a_{-n} = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}}\,dw \text{ for } -n \geq 1$$

and $f_1(a+1/z) = \sum_{n=1}^{\infty} B_n z^n$ for $0 < |z| < 1/R_1$ or

$$\begin{aligned} f_1(z) &= \sum_{n=1}^{\infty} B_n (z-a)^{-n} \text{ for } z \in \text{ann}(a; R_1, \infty) \\ &= \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} = \sum_{n=-\infty}^{-1} a_n (z-a)^n. \end{aligned}$$

So

$$\begin{aligned} f(z) &= f_1(z) + f_2(z) \\ &= \underbrace{\sum_{n=-\infty}^{-1} a_n(z-a)^n}_{\text{for } |z-a|>R_1} + \underbrace{\sum_{n=0}^{\infty} a_n(z-a)^n}_{\text{for } |z-a|<R_2} \\ &= \sum_{n=-\infty}^{\infty} a_n(z-a)^n \text{ for } z \in \text{ann}(a; R_1, R_2) \end{aligned}$$

where the a_n are as claimed. □

Claim. *The Laurent series converges absolutely and uniformly on* $r_1 \leq |a| \leq r_2$.

Since $f_2(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $z \in B(a; R_2)$ then this series converges absolutely for $|z-a| < R_2$ by Theorem III.1.3(a) and converges uniformly for $|z-a| \leq$

r_2 for any $0 < r_2 < R_2$ by Theorem III.1.3(c). Similarly, $g(z) = \sum_{n=1}^{\infty} B_n z^n$ converges absolutely for $|z-a| < 1/R_1$ and converges uniformly for $|z-a| \le 1/r_1$ for any $r_1 > R_1$ by Theorem III.1.3(a) and (c). So $f_1(z) = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$ converges absolutely for $|z-a| > R_1$ and uniformly for $|z-a| \ge r_1$. Therefore $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$ converges absolutely and uniformly on $\mathrm{ann}(a; r_1, r_2)^-$ if $R_1 < r_1 < r_2 < R_2$. □

Claim. *The Laurent series representation is unique for given f.*

Now for the uniqueness. If we were dealing with an analytic function on $B(a; R)$, we could deduce that the coefficients are unique, in fact $a_n = f^{(n)}(a)/n!$ as given in Theorem IV.2.8. For a Laurent series, we integrate. Let $R_1 < r_1 < r_2 < R_2$ and let $\gamma = ((r_1 + r_2)/2)e^{it}$ for $t \in [0, 2\pi]$. Notice that the Laurent series converges uniformly on $\mathrm{ann}(a; r_1, r_2)^-$ as given above. So for $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$ we have for $k \ge 0$:

$$\begin{aligned}
\int_\gamma \frac{f(w)}{(w-a)^{k+1}}\,dw &= \int_\gamma \sum_{n=-\infty}^{\infty} a_n(w-a)^{n-(k+1)}\,dw \\
&= \int_\gamma \left(\sum_{n=-\infty}^{-1} a_n(w-a)^{n-(k+1)} + \sum_{n=0}^{\infty} a_n(w-a)^{n-(k+1)} \right) dw \\
&\qquad \text{since the convergence is absolute—see Definition V.1.10} \\
&= \int_\gamma \left(\sum_{n=-\infty}^{-1} a_n(w-a)^{n-(k+1)} \right) dw + \int_\gamma \left(\sum_{n=0}^{\infty} a_n(w-a)^{n-(k+1)} \right) dw \\
&= \sum_{n=-\infty}^{-1} \left(\int_\gamma a_n(w-a)^{n-(k+1)}\,dw \right) + \sum_{n=0}^{\infty} \left(\int_\gamma a_n(w-a)^{n-(k+1)}\,dw \right) \\
&\qquad \text{since the convergence is uniform; Lemma IV.2.7} \\
&= \sum_{n=1}^{\infty} \left(\int_\gamma \frac{a_{-n}}{(w-a)^{n+k+1}}\,dw \right) + \sum_{n=0}^{\infty} \left(\int_\gamma a_n(w-a)^{n-(k+1)}\,dw \right) \\
&= 0 + 0 + \int_\gamma \frac{a_k}{w-a}\,dw \text{ since each integrand has a primitive on}
\end{aligned}$$

$$G = \text{ann}(a; r_1, r_2) \text{ for } k \neq n$$
$$= a_k 2\pi i\, n(\gamma; a) = a_k 2\pi i.$$

So for $k \geq 0$ it must be that $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{k+1}}\,dw$. The arbitrary nature of γ is explained above. For $k \leq -1$, consider

$$\begin{aligned}
\int_\gamma \frac{f(w)}{(w-a)^{k+1}}\,dw &= \int_\gamma \sum_{n=-\infty}^{\infty} a_n(w-a)^{n-k-1}\,dw \\
&= \int_\gamma \left(\sum_{n=-\infty}^{-1} a_n(w-a)^{n-k-1} + \sum_{n=0}^{\infty} a_n(w-a)^{n-k-1}\right) dw \\
&= \int_\gamma \left(\sum_{n=-\infty}^{-1} a_n(w-a)^{n-k-1}\right) dw + \int_\gamma \left(\sum_{n=0}^{\infty} a_n(w-a)^{n-k-1}\right) dw \\
&= \sum_{n=-\infty}^{-1} \left(\int_\gamma a_n(w-a)^{n-k-1)}\,dw\right) + \sum_{n=0}^{\infty} \left(\int_\gamma a_n(w-a)^{n-k-1}\,dw\right) \\
&= \sum_{n=1}^{\infty} \left(\int_\gamma \frac{a_{-n}}{(w-a)^{n+k+1}}\,dw\right) + \sum_{n=0}^{\infty} \left(\int_\gamma a_n(w-a)^{n-k-1}\,dw\right) \\
&= 0 + \int_\gamma \frac{a_k}{w-a}\,dw + 0 = a_k 2\pi i\, n(\gamma; a) = a_k 2\pi i.
\end{aligned}$$

So for $k \leq -1$ it must be that $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{k+1}}\,dw$. The arbitrary nature of γ is explained above. ∎

Revised: 4/4/2018