Supplement. Applications of the Maximum Modulus Theorem to Polynomials

Note. These notes are a supplement to my online notes on Section 4.54. The Maximum Principle for Complex Variables (MAT 4337/5337) and a supplement to my online notes on Section VI.1. The Maximum Principle for Complex Analysis 2 (MATH 5520).

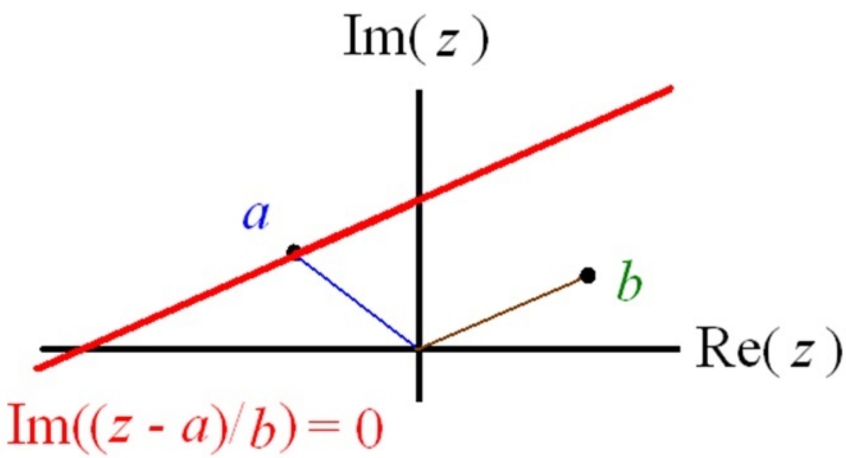
Note. We will prove: The Centroid Theorem, The Lucas Theorem, the Eneström-Kakeya Theorem, a rate of growth theorem, and Bernstein's Theorem.

Note. The next result does not depend on the Maximum Modulus Theorem, but is an interesting result on polynomials and so we include it. The *centroid* of a finite collection of complex numbers, $z_1, z_2, \ldots, z_n$, is simply the arithmetic mean of the numbers, $\sum_{k=1}^{n} z_k/n$.

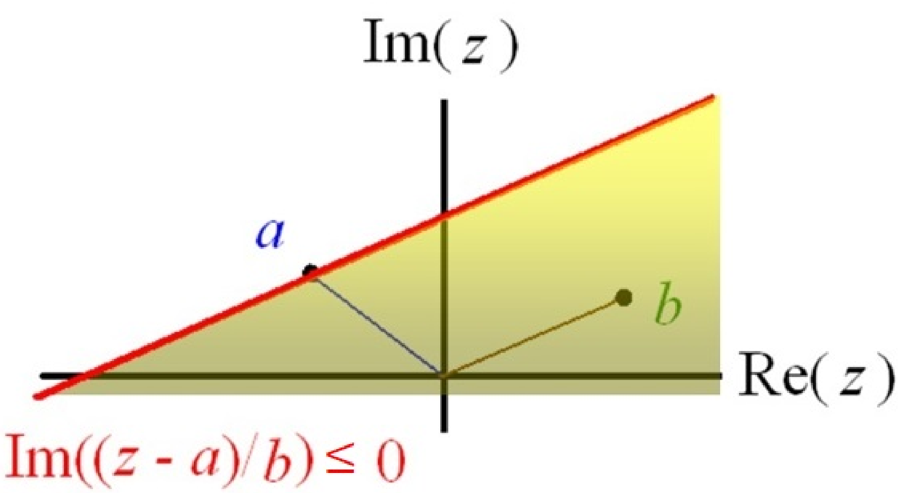
The Centroid Theorem.

The centroid of the zeros of a polynomial is the same as the centroid of the zeros of the derivative of the polynomial.

Note. Also related to the location of zeros of a polynomial is the Lucas Theorem. We need to recall some results concerning the formula of a line in $\mathbb{C}$. In my online notes for Complex Analysis 1 (MATH 5510) on Section I.5. Lines and Half Planes in the Complex Plane, it is argued that a line in the complex plane can be represented by an equation of the form $\text{Im}((z-a)/b) = 0$ where the line is "parallel" to the vector b and translated from the origin by an amount a (here we are knowingly blurring the distinction between vectors in $\mathbb{R}^2$ and numbers in $\mathbb{C}$).



We can represent a closed half-plane with the equation $\text{Im}((z-a)/b) \leq 0$. This represents the half-plane to the right of the line $\text{Im}((z-a)/b) = 0$ when traveling along the line in the "direction" of b.



Note. The proof of the following result does not require the Maximum Modulus Theorem, but does use the Fundamental Theorem of Algebra (the proof of which we have based, ultimately, on Cauchy's Theorem).

The Lucas Theorem [or "Gauss-Lucas Theorem"] (1874).

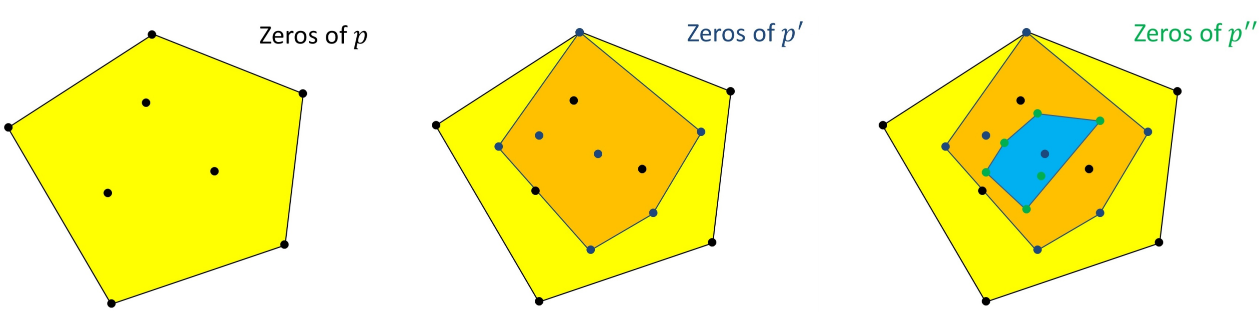
If all the zeros of a polynomial p lie in a half-plane in the complex plane, then all the zeros of the derivative p' lie in the same half-plane.

Note. With repeated application of the Lucas Theorem, we can prove the following corollary.

The Lucas Corollary.

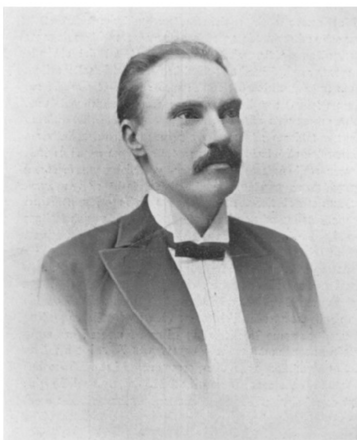
The convex polygon in the complex plane which contains all the zeros of a polynomial p also contains all the zeros of p'.

Note. For example, we might have zeros of p and its derivatives as follows...



Note. The convex polygons above all contain the centroid of the zeros and as we look at higher order derivatives, the polygons close in around the centroid.

Note. Gustav Eneström while studying the theory of pensions was the first to publish a result concerning the location of the zeros of a polynomial with monotone, real, nonnegative coefficients. He published his work in 1893 in Swedish. Sōichi Kakeya published a similar result in 1912 in English.



Gustav Eneström (1852–1923)

Sōichi Kakeya (1886–1947)

The Eneström-Kakeya Theorem.

If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with coefficients satisfying

$$0 \le a_0 \le a_1 \le \cdots \le a_n,$$

then all the zeros of p lie in $|z| \le 1$.

Note. There are MANY generalizations of the classical Eneström-Kakeya Theorem! For a survey of such results, see "The Enestrom-Kakeya Theorem and Some of Its Generalizations," by R. Gardner and N. K. Govil, in *Current Topics in Pure and Computational Complex Analysis*, ed. S. Joshi, M. Dorff, and I. Lahiri, New Delhi: Springer-Verlag (2014), 171–200. Two such results are as follows.

Theorem. (Joyal, Labelle, Rahman 1967).
If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_0 \le a_1 \le \cdots \le a_n$, then all the zeros of p lie in $|z| \le (a_n - a_0 + |a_0|)/|a_n|$.

Theorem. (Gardner, Govil 1994).
If $p(z) = \sum_{j=0}^{n} a_j z^j$, where $\text{Re}(a_j) = \alpha_j$ and $\text{Im}(a_j) = \beta_j$ for $j = 0, 1, 2, \ldots, n$. If

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \text{ and } \beta_0 \le \beta_1 \le \cdots \le \beta_n,$$

then all the zeros of p lie in $|z| \le (|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n))/|a_n|$.

Note. Sergei Bernstein was born in the Ukraine in 1880. He spent time in Göttingen, where he worked under David Hilbert, but did his doctoral work (which involved the solution of one of Hilbert's posed problems) at the Sorbonne in Paris. He taught at Kharkov University (in the Ukraine) from 1907 to 1932. He took a job at the USSR Academy of Scinece in Leningrad in 1932. In 1939 he started working at Moscow University, but kept a home in Leningrad. His son died during the Nazi siege of Leningrad in World War II and after the war he moved to Moscow where he died in 1968. He did fundamental research in probability theory and approximation theory. The "Bernstein's Inequality" which we will state appears in his *Le cons sur les propriétés extrémales et la meilleure approximation de fonctions dex analytiques d'une variable réelle* (Paris, 1926; see Section I.10, "Relations entre le module maximum d'um polynome et celui de ses dérivées sur un segment donné").

Sergei Bernstein (1880–1968)

Note. The first result of Bernstein which we present is based on the Maximum Modulus Theorem.

Rate of Growth Theorem (Bernstein).

If p is a polynomial of degree n such that $|p(z)| \leq M$ on $|z| = 1$, then for $R \geq 1$ we have

$$\max_{|z|=R} |p(z)| \leq MR^n.$$

Note. A great number of generalizations of Bernstein's Rate of Growth Theorem also exist. Here are two.

Theorem. (Ankeny and Rivlin, 1955).
If p is a polynomial of degree n such that $p(z) \neq 0$ for $|z| < 1$ and $|p(z)| \leq M$ on $|z| = 1$, then for $R \geq 1$ we have

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} M.$$

Theorem. (Aziz and Dawood, 1988.)
If p is a polynomial of degree n such that $p(z) \neq 0$ for $|z| < 1$ and $|p(z)| \leq M$ on $|z| = 1$, then for $|z| = R \geq 1$ we have

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} M - \frac{R^n - 1}{2} \min_{|z|=1} |p(z)|.$$

Note. The next result we consider is called "Bernstein's Inequality." It's proof is based on the Maximum Modulus Theorem for Unbounded Domains and the Lucas Theorem.

Definition. For a polynomial p, define the *norm* $\|p\| = \max_{|z|=1} |p(z)|$. This is sometimes called the "sup norm" or "infinity norm" denoted $\|p\|_\infty$.

Note. Bernstein's Inequality in the complex setting states: "If p is a polynomial of degree n, then $\|p'\| \le n\|p\|$. Equality holds if and only if $p(z) = \lambda z^n$ for some $\lambda \in \mathbb{C}$."

Note. Bernstein's original result (in 1926) concerned trigonometric polynomials, which are of the form $\sum_{v=-n}^{n} a_v e^{iv\theta}$. The version presented here is a special case of Bernstein's general result. There is a lengthy history of the so-called "Bernstein's Inequality" (there is also a different result in statistics with the same name). To prove our version, we first need a lemma.

Bernstein Lemma. Let p and q be polynomials such that (i) $\lim_{|z|\to\infty} |p(z)/q(z)| \le 1$, (ii) $|p(z)| \le |q(z)|$ for $|z| = 1$, and (iii) all zeros of q lie in $|z| \le 1$. Then $|p'(z)| \le |q'(z)|$ for $|z| = 1$.

Bernstein's Inequality.

Let p be a polynomial of degree n. Then

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|.$$

Note. There is a rich history of research based on Bernstein's Inequality. For a survey of such results for polynomials, see *Extremal Problems and Inequalities of Markov-Bernstein Type for Algebraic Polynomials*, by R. Gardner, N. K. Govil and G. V. Milovanović, Elsevier Press (2022). Here are two results related to Bernstein's Inequality.

Erdös-Lax Theorem (1944).

Let p be a polynomial of degree n where $p(z) \neq 0$ for $|z| < 1$. Then

$$\|p'(z)\|_\infty \leq \frac{n}{2}\|p(z)\|_\infty,$$

where $\|p(z)\|_\infty = \max_{|z|=1} |p(z)|$.

de Bruijn's Theorem (1947).

Let p be a polynomial of degree n where $p(z) \neq 0$ for $|z| < 1$. Then for $1 \leq \delta \leq \infty$,

$$\|p'\|_\delta \leq \frac{n}{\|1+z\|_\delta}\|p\|_\delta,$$

where $\|p\|_\delta = \left\{\int_0^{2\pi} |p(e^{i\theta})|^\delta \, d\theta\right\}^{1/\delta}$ for $1 \leq \delta < \infty$.

Eneström-Kakeya Theorem References

1. G. Eneström, Härledning af an allmän formel för antalet pensionärer, som vid en godtyeklig tidpuckt förefinnas inom en sluten pensionsclassa. Öfvers, Vetensk.-Akad Förh., **50**, 405–415 (1893).

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4. A. Joyal, G. Labelle, Q. I. Rahman, On the Location of Zeros fo Polynomials, *Canadian Math. Bulletin*, **10**(1), 53–63 (1967).

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Rate of Growth References

1. N. C. Ankeny and T. J. Rivlin, On a Theorem of S. Bernstein, *Pacific Journal of Mathematics*, **5** (1955), 849–852.

2. A. Aziz and M. Dawood, Inequalities for a Polynomial and its Derivative, *Journal of Approximation Theory*, **53** (1988), 155–162.

Bernstein Inequality References

1. S. Bernstein, *Sur L'prdre de la meilleur approximation des fonctions continues par des polynômes de degré donné*, Memoire d el'Académie Royal de Belgique 1912, **4**(2): 1–103.

2. N. de Bruijn, Inequalities Concerning Polynomials in the Complex Domain, *Nederl. Akad. Wetensch. Proc.* **50** (1947), 1265–1272; *Indag. Math.* **9** (1947), 591–598.

3. R. Gardner and N. K. Govil, *Bernstein-Type Inequalities for Polynomials*, Elsevier and Academic Press (2018), to appear.

4. P. D. Lax, Proof of a Conjecture of P. Erdös on the Derivative of a Polynomial, *Bulletin of the American Mathematical Society*, **50** (1944), 509–513.

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