Chapter I. The Complex Number System
Study Guide

The following is a brief list of topics covered in Chapter I of Conway's *Functions of One Complex Variable*, 2nd edition. This list is not meant to be comprehensive, but only gives a list of several important topics. You should also carefully study the proofs given in class and the homework problems.

Section I.1. The Real Number System.
Ring, addition, multiplication, left/right distribution laws, commutative ring, ring with unity, unit, division ring, field, ordering of a field and the Law of Trichotomy, upper/lower bound of a set in an ordered field, least upper bound, complete ordered field, definition of the real numbers, every complete ordered field is isomorphic to $\mathbb{R}$ ("Theorem 2.3.3"), the complex numbers are not ordered (Note 1.1.D), the historical introduction of the complex numbers (Note 1.1.F: Cardano and *Ars Magna*, complex numbers from the quadratic equation, the cubic formula of Tartaglia and its use to find real roots using complex numbers), support and opposition to complex numbers (Note 1.1.G: Albert Girard, Rafael Bombelli, René Descartes).

Section I.2. The Field of the Complex Numbers.
The Cayley-Dickson Construction and "algebras" (Note 1.2.A), history of the definition of $\mathbb{C}$ (Note 1.2.B: Hamilton's definition, the quaternions $\mathbb{H}$, Cayley's definition of the octonions $\mathbb{O}$, John Graves), definition of the complex numbers (addition, multiplication, identities, and inverses), real part, imaginary part, '$\mathbb{C}$ is a two dimensional extension field of $\mathbb{R}$,' '$\mathbb{C}$ is a two dimensional algebra over $\mathbb{R}$,' absolute value/modulus, conjugate, algebraic properties of complex numbers and conjugates (Theorem I.2.A and Corollary I.2.A).

Supplement. Ordering the Complex Numbers.
Ordered field, Law of Trichotomy, inequalities in ordered fields (Theorems 1 and 2), $i \not\in P$ (Corollary 1), $-i \not\in P$ (Corollary 2), proof that $\mathbb{C}$ is not an ordered field (Theorem 3), lexicographic ordering of $\mathbb{C}$, well-ordering, total ordering, comparable, the Well-Ordering Principle.

Section I.3. The Complex Plane.
Caspar Wessel contributions (Note 1.3.A: Wessel's education and employment, Wessel's introduction of the complex plane in 1797, Cristian Juel, Argand's introduction of the complex plane and the "Arand diagram" in 1806), Jean Argand's contributions (Note 1.3.B: Circulation of his work to Legndreand François Français, Jaqcues Français, Argand's other publications), the geometric relationship between $\mathbb{R}^2$ and $\mathbb{C}$ (Note 1.3.C and the ARgand diagram), Parallelogram Law of Addition, norm and metric on $\mathbb{C}$, Triangle Inequality (Theorem I.3.A), equality in the Triangle Inequality (Corollary I.3.A), completeness of $\mathbb{C}$ in terms of Cauchy sequences (Theorem I.3.B and Note 1.3.D), the history of completeness (Note 1.3.E: Cauchy, Bolzano, Dedekind, Dedekind cuts).

Supplement. Location of Zeros of Polynomials
The Fundamental Theorem of Algebra and zeros of a polynomial, the two "categories" of results given in this supplement, Cauchy's Location of Zeros Theorem Category 1 (Theorem 1), an inner radius to the zero containing region due to Cauchy (Corollary 1), Descartes Rule of Signs, Cauchy's Location of Zeros Theorem Category 2 (Theorem 2), Gauss' Location of Zeros Theorem Category 2 (Theorem 3), Hölder's Inequality, Kuniyeda Motel and Tôya's Theorem (Theorem 4), Joyal Labelle Rahman Generalization of Theorem 1 (Theorem 5), an inner radius to the zero containing region due to Joyal Labelle Rahman (Corollary 2), Datt and Govil's Theorem Category 2 (Theorem 6), Datt and Govil's Theorem Category 1 (Theorem 7).

Section I.4. Polar Representation and Roots of Complex Numbers.
History of polar coordinates (Note I.4.A: Bonaventura Cavalieri, Gregorius Saint-Vincent, Isaaac Newton, Jacob Bernoulli), an argument of a complex number $\arg(z)$, "$\mathrm{cis}(\theta)$," multiplication of complex numbers in polar form (Theorem 1.4.A), de Moivre's Formula (Corollary I.4.A), de Moivre's contributions (Note 1.4.B: His biography, Cardan's formula, Euler's use of the formula), n^{th} roots of a complex number (in particular, n^{th} roots of unity.

Section I.5. Lines and Half Planes in the Complex Plane.
The parametric representation of a line in Calculus 3, equation of a line in $\mathbb{C}$, "direction vector" and "translation vector" of a line in $\mathbb{C}$, half-plane in $\mathbb{C}$.

Supplement. The Ilief-Sendov Conjecture.
Lines and half-planes in $\mathbb{C}$, the Gauss-Lucas Theorem (Theorem 1) its proof and corollary, the Fundamental Theorem of Algebra, a circle which contains all of the zeros of polynomial P also contains all of the zeros of P' (Corollary 2 and Corollary 3), the centroid of zeros of a polynomial, the centroid of the zeros of a polynomial P is the same as the centroid of the zeros of P' (Theorem 2) and its proof, the Ilieff-Sendov Conjecture, specila cases of the Ilieff-Sendov Conjecture which have been proved, the Goodman-Rahman-Ratti Conjecture, Sofi Ahanger and Gardner's Theorem (Theorem A), Sofi and Shah's Theorem (Theorem B), Terence Tao's manuscript and claim (Theorem C), Petar Petrov's claimed proof ("Breaking News!").

Supplement. Dr. Bob's Favorite Results on Complex Polynomials.
Analytic function, Cauchy's theorem relating the nth derivative and an integral, Cauchy's Inequality, Liouville's Theorem, violation of Liouville's Theorem by real functions, the Maximum Modulus Theorem, the Fundamental Theorem of Algebra and its proof based on Liouville's Theorem, the Centroid Theorem and its proof, lines and half-planes in $\mathbb{C}$, the Lucas Theorem (or Gauss-Lucas Theorem) its proof and corollary, Gustav Eneström, Sōichi Kakeya, the Eneström-Kakeya Theorem and its proof, results related to the Eneströ-Kakeya Theorem (Joyal Labelle Rahman 1967, and Gardner Govil 1994), Jensen's Formula, Titchmarsh's Number of Zeros Theorem and its proof, Pukhta's 2011 number of zeros theorem, Bernstein's Rate of Growth Theorem and its proof, other rate of growth results (Ankeny and Rivlin 1955, and Aziz and Dawood 1988), the norm of a polynomial, trigonometric polynomials, Bernstein's Inequality and its proof, results related to Bernstein's

Inequality (Erdős-Lax Theorem 1944, de Bruijn's Theorem 1947), the Iliev-Sendov Conjecture and its introduction, special cases of the Iliev-Sendov Conjecture which have been proved, Goodman-Rahman-Ratti Conjecture and a counterexample.

Section I.6. The Extended Plane and its Spherical Representation.
Metirc, sxtended complex plane $\mathbb{C}_\infty$, meromorphic functions, the Riemann sphere, relationships between (x_1, x_2, x_3) and z, stereographic projections, stereographic projections and graph theory, the metric on $\mathbb{C}_\infty$, the history of stereographic projection (the Egyptians and Hipparchus likely knew of it, Claudius Ptolemy gives the earliest known account in his *Planisphaerium* in which he projected through the "south pole").

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