The Stability Theorem for Orders of Zeros of Equations

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The general idea of the *stability* of a system is that the system can be slightly perturbed and it will return to its original state. For example, a ball at the bottom of a well (where it is at equilibrium and not rolling around) can be slightly perturbed and it will roll back to its original position. A ball on the top of a hill will be in equilibrium (not rolling around), but if it is slightly perturbed, then it will roll down the hill and not return to its original position.

The "Stability Theorem for Orders of Zeros of Equations" is a result dealing with a similar situation. It says, informally, that if an equation has m solutions and the equation is slightly perturbed, then the perturbed equation also has m solutions. You should be warned that the title of this theorem is due to Dr. Bob and is descriptive of the result, but probably not used by anyone else! We need a couple of results from the exercises to give a complete proof. All results are from *Functions of One Complex Variable*, Second Edition, by John Conway, NY: Springer-Verlag, 1978.

Lemma 1. (Exercise IV.7.1) If $f : G \to \mathbb{C}$ is analytic and γ is a rectifiable curve in G then $f \circ \gamma$ is also a rectifiable curve.

Lemma 2. (Exercise IV.7.3) Let f be analytic in $B(a;R)$ and suppose that $f(a) = 0$. Then a is a zero of multiplicity m if and only if $f^{(m-1)}(a) = f^{(m-2)}(a) = \cdots = f'(a) = f(a) = 0$ and $f^{(m)}(a) \neq 0$.

Partial Proof. For a a zero of multiplicity m, we know that $f(z) = (z - a)^m g(z)$ where g is analytic in $B(a;R)$ and $g(a) \neq 0$ (by the definition of "zero of multiplicity m"). One can show by induction that

$$f^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} \binom{m}{k} (z-a)^{m-k} g^{(n-k)}(z)$$

for $n = 0, 1, 2, \ldots, m$. So $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \ldots, m-1$ and $f^{(m)}(a) = g^{(0)}(a) = g(a) \neq 0$. $\square$

Theorem. The Stability Theorem for Orders of Zeros of Equations. (Theorem IV.7.4) Suppose f is analytic in $B(a;R)$ and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at $z = a$ then there is an $\varepsilon > 0$ and $\delta > 0$ such that for $0 < |\zeta - \alpha| < \delta$, the equation $f(z) = \zeta$ has exactly m simple roots in $B(a, \varepsilon)$.

Proof. Since the zeros of an analytic function are isolated (by Corollary IV.3.10), we can choose $\varepsilon > 0$ such that

1. $\varepsilon < R/2$,
2. $f(z) = \alpha$ has no solutions in $0 < |z - a| < 2\varepsilon$ (since the zeros of f are isolated), and
3. $f'(z) \neq 0$ for $0 < |z - a| < 2\varepsilon$ (if $m = 1$ then $f'(a) \neq 0$ and so f' is nonzero 'near' a since f' is continuous; if $m \geq 2$ then $f'(a) = 0$ and the zeros of f' are isolated).

Let $\gamma(t) = a + \varepsilon \exp(2\pi i t)$ where $t \in [0, 1]$ and define $\sigma = f \circ \gamma$. Notice that σ is rectifiable by Lemma 1. Now $\alpha \notin \{\sigma\}$ since $f(z) \neq \alpha$ on $\{\gamma\}$. So there is a $\delta > 0$ such that $B(a;\delta) \cap \{\sigma\} = \emptyset$. Thus, $B(a;\delta)$ is contained in some component of $\mathbb{C} \setminus \{\sigma\}$. That is, $|\alpha - \zeta| < \delta$ implies $n(\sigma;\alpha) = n(\sigma;\zeta)$ by Theorem IV.4.4. Now

$$n(\sigma;\alpha) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w-\alpha} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz = \sum_{k=1}^{m} n(\gamma, a_k) = m$$

where $a_1, a_2, \ldots, a_m$ are the values of z such that $f(z) = \alpha$ (so $a_1 = a_2 = \cdots = a_m = a$). [Notice that this computation allows us to transition from winding numbers involving σ, which we may not know much about, to winding numbers involving γ, and $\{\gamma\}$ is just a circle.] Next,

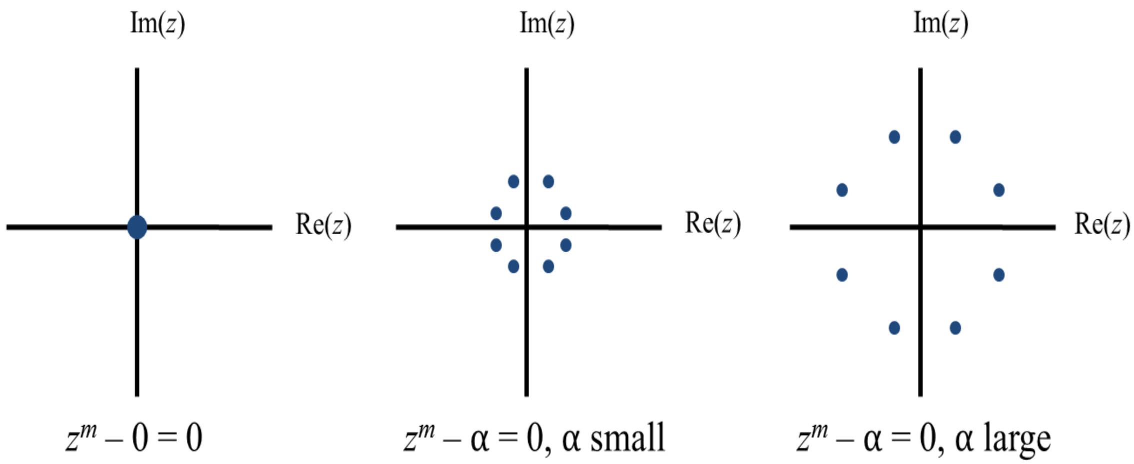
$$n(\sigma;\zeta) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w-\zeta} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\zeta}\,dz = \sum_{k=1}^{p} n(\gamma, z_k(\zeta))$$

where $z_1(\zeta), z_2(\zeta), \ldots, z_p(\zeta)$ are the values of z such that $f(z) = \zeta$. Therefore

$$m = n(\gamma;\alpha) = n(\sigma;\zeta) = \sum_{k=1}^{p} n(\gamma; z_k(\zeta)).$$

Now $n(\gamma; z_k(\zeta))$ must be either 0 or 1 since γ is simply a positively oriented circle. So there are m values of z in $B(a,\varepsilon)$ such that $f(z) = \zeta$. Since $f'(z) \neq 0$ for $0 < |z-a| < \varepsilon$, each of these roots (for $\zeta \neq \alpha$, which is a hypothesis) must be simple by Lemma 2. ∎

Note. To illustrate this result, consider the function $f(z) = z^m$ and $\alpha = 0$. Then $a = 0$ is a zero of order m for the equation $z^m - 0 = 0$. If we take ζ "near" $\alpha = 0$ and consider the perturbed equation $f(z) - \zeta = 0$, or equivalently $z^m - \zeta = 0$, then we see that there are m simple roots, namely the m roots of ζ. This is illustrated below.



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