Supplement. Transformations of $\mathbb{C}$ and $\mathbb{C}_\infty$—An Approach to Geometry

Note. We now supplement Section III.3 "Analytic Functions as Mappings: Möbius Transformations" of Conway with a related topic on geometry from *Geometry with an Introduction to Cosmic Topology* by Michael Hitchman, Boston: Jones and Bartlett Publishers (2009). We first explore transformations in general, define *geometry* in a transformation setting, and finally use Möbius transformations to explore hyperbolic geometry. When quoting from Hitchman, we use his numbering scheme prefaced with an "H."

Definition H.3.1.1. A *transformation* on set A is a function $T : A \to A$ that is one to one and onto.

Note. A transformation $T : A \to A$ has an *inverse* $T^{-1} : A \to A$ such that $T(T^{-1}(x)) = T^{-1}(T(x)) = x$. If S and T are translations on set A then the compositions $S \circ T$ and $T \circ S$ are also transformations on A.

Example H.3.1.6. The transformation $T(z) = kz$ where $k > 0$ is a *stretch transformation* of $\mathbb{C}$. If $k > 1$ then it "stretches" point z to point kz (along a line passing through 0 and z). If $0 < k < 1$ then it "shrinks" point z to kz (along a line passing through 0 and z).

Example H.3.1.7. Let $a, b \in \mathbb{C}$, $a \neq 0$. Then $T(z) = az + b$ is a *general linear transformation* of $\mathbb{C}$.

Note. General linear transformations are combinations of translations (T_b), rotations ($R_{\arg(a)}$), and stretches (by an amount $k = |a|$):

$$T(z) = az + b = \left(e^{i\arg(a)}(|a|z)\right) + b.$$

Note. We will approach geometry by finding transformations which preserve "geometrical objects" (such as lines and circles). This was the approach of Felix Klein (1849–1925) who started with a group of transformations on a set and then "threw out all concepts that did not remain unchanged under these transformations. ...Klein's approach to geometry, called the *Erlangen Program* after the university at which he worked at the time, has the benefit that all three geometries (Euclidean, hyperbolic, and elliptic) emerge as special cases from a general space and a general set of transformations" [Hitchman, page 6]. For these reasons, we are interested in lines and circles.

Recall. A line in $\mathbb{C}$ containing point a with direction $b \neq 0$ is [Conway page 6] $\mathrm{Im}((z-a)/b) = 0$. We can write this as

$$\begin{aligned} \mathrm{Im}\left(\frac{z-a}{b}\right) &= \mathrm{Im}\left(\frac{z}{z}\right) + \mathrm{Im}\left(\frac{-a}{b}\right) \\ &= \frac{z/a + \overline{z}/\overline{b}}{2i} + \mathrm{Im}\left(\frac{-a}{b}\right) \end{aligned}$$

$$= \frac{z}{2bi} + \frac{\overline{z}}{-2\overline{b}i} + \text{Im}\left(\frac{-a}{b}\right)$$
$$= \alpha z + \overline{\alpha}\,\overline{z} + d = 0$$

where $\alpha = 1/(2bi)$ and $d = \text{Im}(-a/b) \in \mathbb{R}$. So a line in $\mathbb{C}$ can be written in the form $\alpha z + \overline{\alpha}\,\overline{z} + d = 0$ for some $\alpha \in \mathbb{C}$ and $d \in \mathbb{R}$.

Recall. A circle C in $\mathbb{C}$ with center z_0 and radius r is the set of all $z \in \mathbb{C}$ such that $|z - z_0| = r$ or $(z - z_0)(\overline{z} - \overline{z}_0) = r^2$.

Theorem H.3.1.9. Suppose T is a general linear transformation. Then T maps lines to lines and T maps circles to circles.

Definition III.3.1. A *path* (or as Hitchman says, a *curve*) in a region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a, b] \to G$ for some interval $[a, b] \subset \mathbb{R}$. If $\gamma'(t)$ exists and is continuous then γ is a *smooth path*. γ is *piecewise smooth* if for $a = t_0 < t_1 < \cdots < t_n = b$, γ is smooth on $[t_{j-1}, t_k]$ for $j = 1, 2, \ldots, n$.

Note. We can associate a vector in $\mathbb{R}^2$ with each element of $\mathbb{C}$. If γ is a smooth path, then $\gamma'(t)$ represents the vector tangent to γ in the direction of increasing t.

Definition. Suppose γ_1 and γ_2 are smooth paths and $\gamma_1(t_1) = \gamma_2(t_2) = a_0$ and $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$. Then an *angle between paths* γ_1 and γ_2 at z_0 is $\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1))$.

Notice. With appropriate interpretation of "arg(z)" we can say $\arg(z_1 z_2) = \arg(z_1)+\arg(z_2)$ [Conway, Page 5] and $\arg(z_1/z_2) = \arg(z_1)-\arg(z_2)$ for all nonzero a_1 and z_2.

Theorem H.3.1.11. General linear transformations preserve angles between intersecting lines.

Definition H.3.1.13. A *Euclidean isometry* is a transformation T of $\mathbb{C}$ that preserves the Euclidean distance between two points: $|T(z) - T(w)| = |z - w|$.

Note. Time permitting, we will define hyperbolic isometry and elliptic isometry later.

Example H.3.1.14. Rotations and translations are Euclidean isometries of $\mathbb{C}$. Therefore a general linear transformation of the form $T(z) = az + b$ where $|a| = 1$ is a Euclidean isometry.

Example H.3.1.15. Reflections about the axis (or the "x-axis") is obtained with the transformation $z \to \overline{z}$. In fact, any reflection about a line can be obtained through translations, rotations, and conjugation. Reflection about the line $y = mx + b$ (where $x, y, m, b \in \mathbb{R}$) is obtained as a transformation of $\mathbb{C}$ to $\mathbb{C}$ as$R_L(z) =$

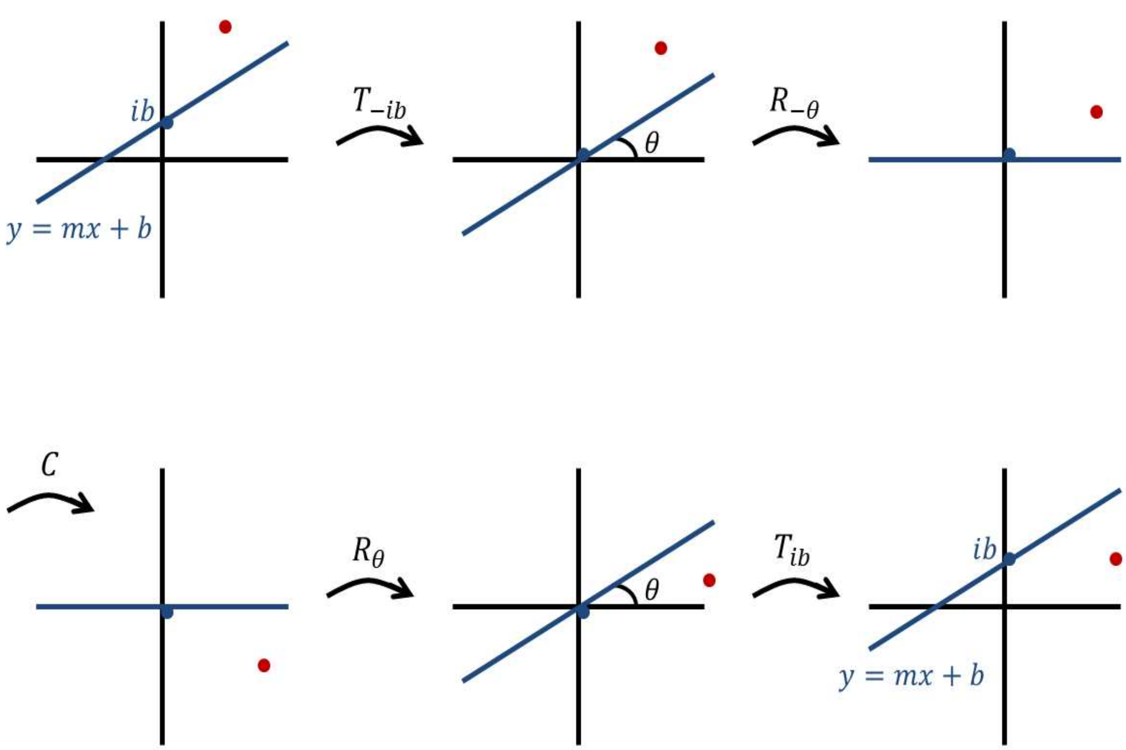
$e^{2i\theta}\overline{z} + ib(e^{2i\theta}+1)$ where $\theta = \tan^{-1}(m)$. Notice that

$$\begin{aligned} R_L(z) &= e^{2i\theta}(\overline{z}+ib)+ib \\ &= e^{i\theta}\left(\overline{e^{-i\theta}(z-ib)}\right)+ib \\ &= T_{ib}(R_\theta(C(R_{-\theta}(T_{-ib})))) \end{aligned}$$

where

$$\begin{aligned} T_{-ib}(z) &= z-ib \text{ (translation by } -ib) \\ R_{-\theta}(z) &= e^{-i\theta}z \text{ (rotation by } -\theta) \\ C(z) &= \overline{z} \text{ (conjugation)} \\ R_\theta(z) &= e^{i\theta}z \text{ (rotation by } \theta) \\ T_{ib}(z) &= z+ib \text{ (translation by } ib) \end{aligned}$$

Geometrically, this is:



Note. Reflections are fundamental transformations in the current setting (rotations, translations, and isometries). Some relevant results illustrating this are:

Theorem H.3.1.16. A *translation* of $\mathbb{C}$ is the composition of reflections about two parallel lines. A *rotation* of $\mathbb{C}$ about a point z_0 is the composition of reflections about two lines that intersect z_0.

Theorem H.3.1.17. Reflection across a line is a Euclidean isometry. Moreover, any reflection sends lines to lines, sends circles to circles, and preserves angle magnitudes.

Theorem H.3.1.18. Any Euclidean isometry is the composition of, at most, three reflections.

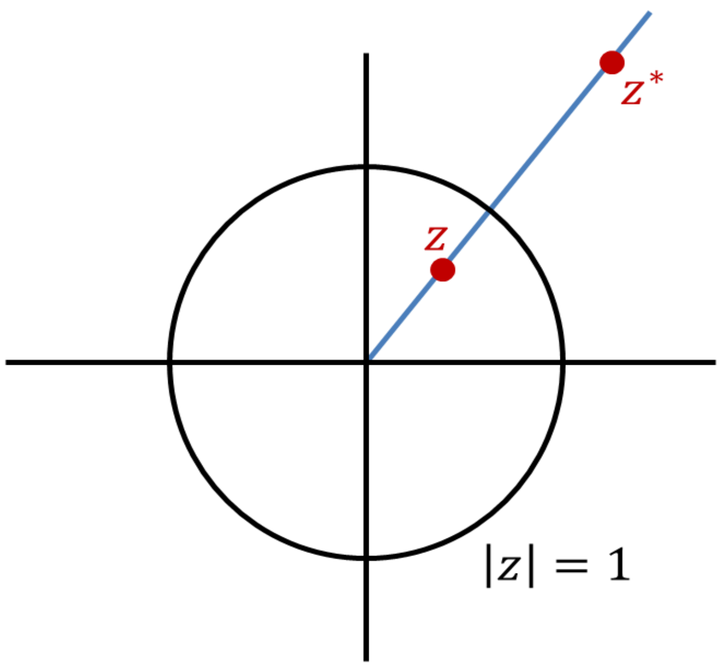
Note. The proof of Theorem H.3.1.18 can be found in *Geometry on Surfaces* by J. Stillwell, NY: Springer (1992).

Note. There is one additional "fundamental transformation" of interest to the study of geometry. It is the *inversion with respect to a cline.* We have seen inversion (i.e., *reflection*) with respect to a line. We will explore reflection with respect to a circle and see that $T(z) = 1/\overline{z}$ is reflection with respect to the unit circle.

Note. $T(z) = 1/\overline{z}$ fixes the points of the unit circle: $T(e^{i\theta}) = 1/\left(\overline{e^{i\theta}}\right) = 1/e^{-i\theta} = e^{i\theta}$. For $z = re^{i\theta} \neq 0$, we have

$$T(z) = T(re^{i\theta}) = \frac{1}{\overline{re^{i\theta}}} = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta} \equiv z^*.$$

The points z and z^* are *symmetric* with respect to the unit circle. Geometrically, z and z^* are colinear with 0 and lie on the same side of 0, and $|z| = 1/|z^*|$:



Definition. Points z and z^* are *symmetric with respect to circle* C where C has center z_0 and radius r if z and z^* are colinear with the center of C (point z_0), lie on the same side of z_0, and $|z - z_0||z^* - z_0| = r^2$.

Note. The geometric mean of the distance of z from z_0 and the distance of z^* from z_0 is r.

Note. Since z_0, z, and z^* are colinear and z and z^* are on the same side of z_0, then $z^* - z_0 = k(z - z_0)$ for some positive k. So

$$k|z - z_0| = |z^* - z_0| = \frac{r^2}{|z - z_0|} \Longrightarrow k = \frac{r^2}{|z - z_0|^2}$$

and

$$z^* - z_0 = k(z - z_0) = (z - z_0)\frac{r^2}{|z - z_0|^2} = \frac{(z - z_0)r^2}{(z - z_0)\overline{(z - z_0)}} = \frac{r^2}{\overline{(z - z_0)}}.$$

Therefore $z^* = \dfrac{r^2}{\overline{(z - z_0)}} + z_0$.

Definition (Example H.3.2.1). Let C be a circle with center z_0 and radius r. Define *H-inversion with respect to circle* C as

$$i_C(z) = \frac{r^2}{\overline{(z - z_0)}} + z_0.$$

If L is a line $y = mx + b$ then define *H-inversion with respect to line* L as in Example H.3.1.15:

$$i_L(z) = e^{2i\theta}\overline{z} + ib(e^{2i\theta} + 1)$$

where $\theta = \tan^{-1}(m)$.

Note. $i_C(z)$ (where C is a circle with center z_0) is not a transformation on $\mathbb{C}$ since it is undefined at z_0. Soon, we will see that $i_C(z)$ is a transformation on $\mathbb{C}_\infty$.

Note. One way to visualize functions of a complex variable is to see how certain sets in the domain ("z-values") are mapped to the range ("w-values"). For inversion, we get something like this:

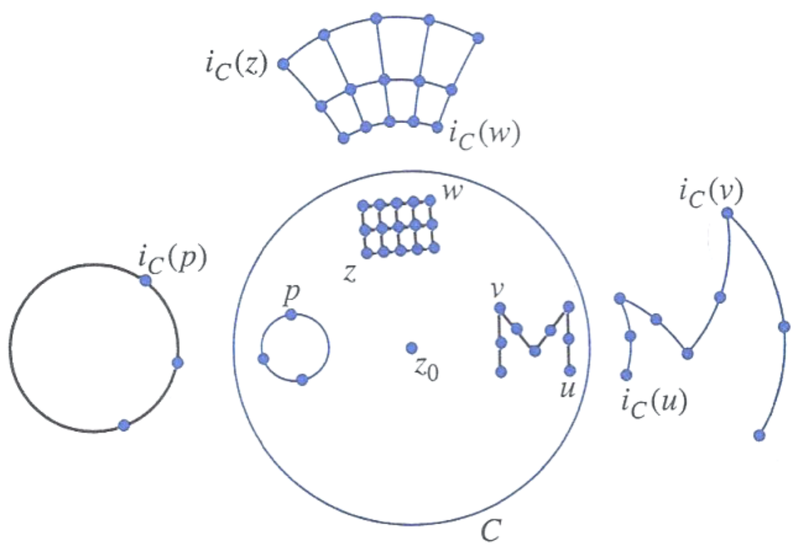


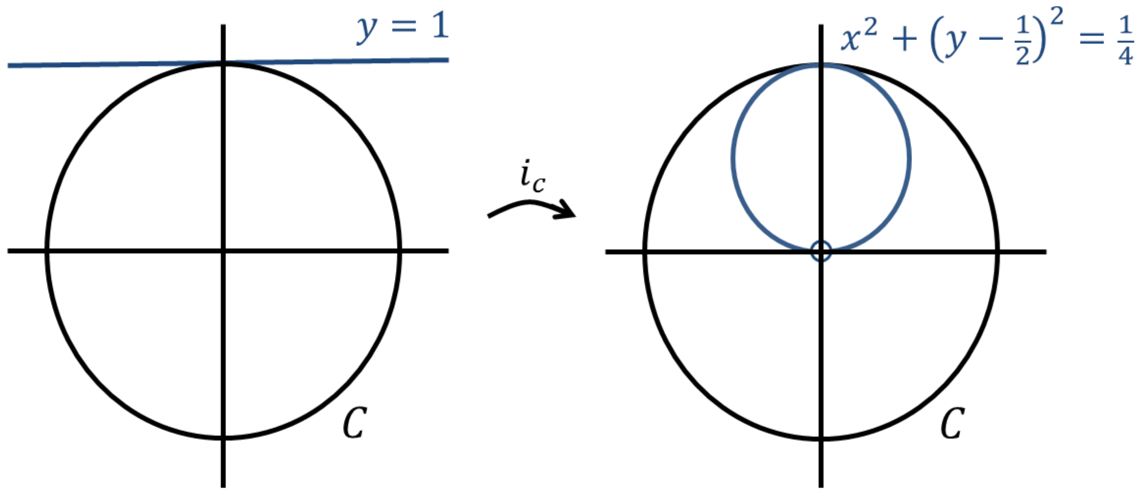
Figure 3.4 from Hitchman.

Note. There are "applets" online that help illustrate various complex functions, including inversions.

Example. Consider the line $y = 1$ (or $\{z \mid \text{Im}(z) = 1\} = \{z \mid z = t + i, t \in \mathbb{R}\}$). Applying H-inversion of the line with respect to the unit circle we get:

$$i_C(z) = i_C(t+i) = \frac{1}{\overline{(t+i)}} = \frac{1}{t-i} = \frac{1}{t-i}\frac{t+i}{t+i} = \frac{t+i}{t^2+1} = \frac{t}{t^2+1} + i\frac{1}{t^2+1}.$$

Let $x = \frac{t}{t^2+1}$ and $y = \frac{1}{t^2+1}$. Then we have $x^2 + (y - 1/2)^2 = 1/4$ and $y \neq 0$. Geometrically:



So H-inversion can map lines to circles (well, "almost circles"). This turns out to be more important (and convenient) than we might suppose.

Definition H.3.2.2. A *cline* is a (Euclidean) line or a circle.

Theorem H-Page 41. Any cline can be described by an equation of the form $cz\overline{z} + \alpha z + \overline{\alpha}\,\overline{z} + d = 0$ where $\alpha \in \mathbb{C}$ is constant and $c, d \in \mathbb{R}$ are constant.

Theorem (Construction H-Page 42). Three distinct complex numbers determine a unique cline.

Theorem H.3.2.4. H-Inversion in a cline maps clines to clines. In particular, if a cline goes through the center of a circle of H-inversion, its image will be a line; otherwise the image of a cline will be a circle.

Note. Some other important properties (from our non-Euclidean geometry perspective) of inversion include:

Theorem H.3.2.5. Suppose C is a circle in $\mathbb{C}$ centered at $z - 0$, and $z \neq z_0$ is not on C. A cline through z is orthogonal to C if and only if it goes through z^*, the point symmetric to z with respect to C.

Theorem H.3.2.6. H-inversion in C takes clines orthogonal to C to themselves (i.e., such clines are fixed by the inversion).

Theorem H.3.2.7. H-inversion is a cline preserves angle magnitudes.

Theorem H.3.2.8. Let i_C denote H-inversion in a cline C. If p and q are symmetric with respect to cline D, then $i_C(p)$ and $i_C(q)$ are symmetric with respect to cline $i_C(D)$. That is, H-inversion preserves symmetric points.

Note. We now return to the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. We can then extend general linear transformations $T(z) = az + b$ to the set $\mathbb{C}_\infty$ by defining $T(\infty) = \infty$. More interestingly, we can extend inversion as follows:

$$i_C(z) = \begin{cases} \frac{r^2}{\overline{(z - z_0)}} + z_0 & \text{if } a \notin \{z_0, \infty\} \\ \infty & \text{if } z = z_0 \\ z_0 & \text{if } z = \infty \end{cases}$$

where C is a circle with center z_0 and radius r.

Note. Conway defines an *inversion* as the swapping $i(z) = z/z$. Notice that this type of inversion is a composition of two H-inversions:

$$i(z) = \frac{1}{z} = \overline{\left(\frac{1}{\overline{z}}\right)} = i_L(i_C(z)).$$

Where $i_K(z) = \overline{z}$ is H-inversion with respect to the x-axis and $i_C(z) = 1/\overline{z}$ is H-inversion with respect to the unit circle.

Note. Before we jump into the topic of Möbius transformations, we motivate them by addressing Felix Klein's approach to geometry. To do so, we need a set (of "points") and a *group* of transformations defined on the set.

Definition H.4.1.3. Let S be any set, and G a group of transformations on S. The pair (S, G) is called a *geometry*. A *figure* in the geometry is any subset A of S. Two figures A and B are *congruent*, denoted $A \cong B$, if there exists a transformation $T \in G$ such that $T(A) = B$.

Example H.4.1.4. the group $\mathcal{T}$ of all transformations on $\mathbb{C}$, $\mathcal{T} = \{T \mid T(z) = z + b, b \in \mathbb{C}\}$ determines *translational geometry* on $\mathbb{C}$, $(\mathbb{C}, \mathcal{T})$. Parallel lines and circles of the same radius are congruent in this geometry. Non-parallel lines and circles of different radii are not congruent.

Note. Hitchman in his Example H.4.1.8 defines *Euclidean geometry* to be $(\mathbb{C}, \mathcal{E})$ where $\mathcal{E}$ consists of all general linear transformations of the following form:

$$\mathcal{E} = \{T \mid T(z) = e^{i\theta}z + b,\ \theta \in \mathbb{R},\ b \in \mathbb{C}\}.$$

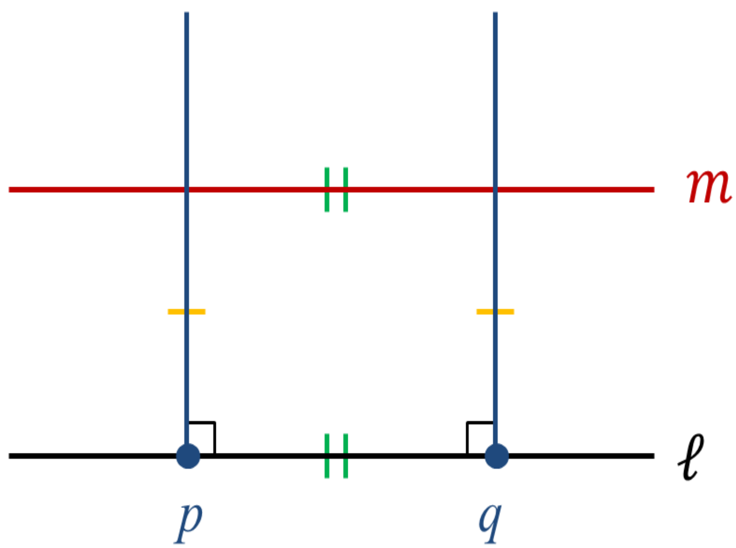
However, this does not include reflections, so that the following triangles would then not be considered congruent (in violation of "side-side-side"):

According to *Elementary Geometry* by Ilka Agricola and Thomas Friedrich, American Mathematical Society, Student Mathematical Library, Volume 43 (2007), Euclidean geometry includes all Euclidean isometries. These isometries on $\mathbb{R}^2$ are of the form $T(\vec{v}) = A\vec{v} + \vec{b}$ where $\vec{b} \in \mathbb{R}^2$ and A is an orthogonal/orthonormal matrix (and the same for Euclidean geometry on $\mathbb{R}^n$). With $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ we map $(x, y) = x + iy$ to $(x, -y) = x - iy$, so reflections are included in this definition.

Theorem H-Page 79. Euclidean distance, $d(z_1, z_2) = |z_1 - z_2|$, is invariant in Euclidean geometry:

$$|T(z_1) - T(z_2)| = |(e^{i\theta}z_1 + b) - (e^{i\theta}z_2 + b)| = |z_1 - z_2|.$$

Note. In Euclidean geometry, parallel lines are equidistant apart. This can be explained in the setting of Book I of *The Elements of Geometry* in terms of parallelograms. That is, let ℓ and m be lines and p and q points on ℓ (not on m). Then two lines perpendicular to ℓ through points p and q will intersect line m in such a way as to determine a rectangle if and only if ℓ and m are parallel:



A proof will follow from, say, the Parallel Postulate (Postulate 5) and Proposition 33 ("Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.") Since Euclidean distance is an invariant of Euclidean geometry, we can *prove* that parallel lines are equidistant from one another (in the transformation setting) and therefore prove the Parallel Postulate in the transformation setting.

Definition H.4.1.9. A geometry $(\mathbb{C}, G)$ is called *homogeneous* if any two points in $\mathbb{C}$ are congruent, and *isotropic* if the transformation group contains rotations about each point in $\mathbb{C}$.

Example H.4.1.10. Translation geometry on $\mathbb{C}$ is homogeneous but not isotropic. Euclidean geometry on $\mathbb{C}$ is homogeneous and isotropic.

Note. We now return to Conway and study Möbius transformations. This collection of transformations form a group of transformations on $\mathbb{C}_\infty$. This "Möbius geometry" is very general (like "neutral geometry" which is an axiomatic approach to geometry that does not have a parallel postulate). Certain subgroups of the Möbius transformations will determine (1) elliptic geometry, (2) Euclidean geometry, and (3) hyperbolic geometry, on subsets of $\mathbb{C}_\infty$.

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