Chapter V. Singularities

V.1. Classification of Singularities

Note. In this section, we define various types of singularities of a function and develop the idea of a Laurent series.

Definition. A function f has an *isolated singularity* at $z = a$ if there is an $R > 0$ such that f is defined and analytic in $B(a; R) \setminus \{a\}$, but not in $B(a; R)$. Point a is a *removable singularity* if there is an analytic function $g : B(a; R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$.

Example. Functions $f_1(z) = 1/z$ and $f_2(z) = \sin z/z$, and $f_3(z) = \exp(1/z)$ each have isolated singularities at $z = 0$. As shown in Exercise V.1.1, $f_2(z) = \sin z/z$ has a removable singularity.

Theorem V.1.2. If f has an isolated singularity at a then $z = a$ is a removable singularity if and only if $\lim_{z \to a}(z - a)f(z) = 0$.

Definition. If $z = a$ is an isolated singularity of f then a is a *pole* of f if $\lim_{z \to a} |f(z)| = \infty$. If an isolated singularity is neither a pole nor a removable singularity it is called an *essential singularity*.

Example. Function $f(z) = \dfrac{1}{(z-a)^m}$ for $m \in \mathbb{N}$ has a pole at $z = a$. Function $g(z) = \exp(z^{-1})$ has an essential singularity at $z = 0$. In fact, a function with a pole at $z = a$ has a well defined form, as given next.

Proposition V.1.4. If G is a region with $a \in G$, and if f is analytic in $G \setminus \{a\}$ with a pole at $z = a$, then there is a positive integer m and an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \dfrac{g(z)}{(z-a)^m}$.

Definition. If f has a pole at $z = a$ and m is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z = a$, then f has a *pole of order* m at $z = a$. A pole of order 1 is called a *simple pole*.

Note. If f has a pole of order m at $z = a$, then $f(z) = g(z)/(z-a)^m$ where g is analytic in $B(a; R)$ (for some $R > 0$), so

$$g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k (z-a)^k$$

and

$$f(z) = \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g_1(z)$$

where g_1 is analytic in $B(a; R)$ and $A_m \neq 0$.

Definition. If f has a pole of order m at $z = a$ and f satisfies

$$f(z) = \frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)} + g_1(z)$$

then

$$\frac{A_m}{(z-a)^m} + \frac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{A_1}{(z-a)}$$

is called the *singular part* of f at $z = a$.

Note. We will see that an essential singularity behaves rather like a pole of infinite order. This then produces an infinite singular part. First, some definitions.

Definition V.1.10. If $\{z_n \mid n \in \mathbb{Z}\}$ is a doubly infinite sequence of complex numbers, then $\sum_{n=-\infty}^{\infty} a_n$ is *absolutely convergent* if both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ are absolutely convergent. If these series are absolutely convergent then define

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n.$$

If u_n is a function on a set S for $n \in \mathbb{Z}$ and $\sum_{n=-\infty}^{\infty} u_n(s)$ is absolutely convergent for every $s \in S$, then the convergence is *uniform* over S if both $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_{-n}$ converge uniformly on S.

Definition. If $0 \le R_1 < R_2 \le \infty$ and a is any complex number, define

$$\text{ann}(a; R_1, R_2) = \{z \mid R_1 < |z-a| < R_2\}.$$

Note. We now deal with a series representation of a function analytic on an annulus.

Theorem V.1.11. Laurent Series Development.
Let f be analytic in $\text{ann}(a; R_1, R_2)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where the convergence is absolute and uniform over the closure of $\text{ann}(a; r_1, r_2)$ if $R_1 < r_1 < r_2 < R_2$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad (1.12)$$

where γ is the circle $|z-a| = r$ for any r with $R_1 < r < R_2$. Moreover, this series is unique.

Note. The proof of Theorem V.1.11 is in a, sort of, self contained supplement. The Laurent series allows us to classify isolated singularities.

Corollary V.1.18. Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ be its Laurent expansion in $\text{ann}(a; 0, R)$. Then

(a) $z = a$ is a removable singularity if and only if $a_n = 0$ for $n \leq -1$,

(b) $a = z$ is a pole of order m if and only if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$, and

(c) $z = a$ is an essential singularity if and only if $a_n \neq 0$ for infinitely many negative integers n.

Note. If f has an essential singularity at $z = a$, then $\lim_{z \to a} |f(z)|$ does not exist. The text says: "This means that as z approaches a the values of $f(z)$ must wander through $\mathbb{C}$." The following result shows that this wandering is very intense.

Theorem V.1.21. Casorati-Weierstrass Theorem.
If f has an essential singularity at $z = a$ then for every $\delta > 0$, $\{f(\text{ann}(a; 0, \delta)\}^- = \mathbb{C}$.

Note. A more general result concerning the behavior of f near an essential singularity is in Chapter XII (the last chapter of the text, page 300):

Great Picard Theorem.
Suppose an analytic function has an essential singularity at $z = a$. Then in each neighborhood of a, f assumes each complex number, with one possible exception, an infinite number of times.

Note. Function $f(z) = \exp(1/z)$ is such a function, and it clearly does not take on the value 0.

Note. For the record, from page 297 we have:

Little Picard Theorem.
If f is an entire function that omits two values, then f is a constant.

Note. Of course, $f(z) = e^z$ is an example of a function omitting one value.

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