V.2. Residues

Note. Given our previous experience with integrals over closed and rectifiable curves, we expect lots of integrals to be 0, except those related to $1/(z-a)$. Hence, in a Laurent expansion, our attention is drawn to a_{-1}. In this section, we also develop some techniques with which we can evaluate integrals of functions of a real variable.

Definition V.2.1. Let f have an isolated singularity at $z = a$ and let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$ be its Laurent expansion about $z = a$. The *residue* of f at $z = a$ is the coefficient a_{-1}, denoted $\text{Res}(f; a) = a_{-1}$.

Note. The following relates residues to integrals and winding numbers.

Theorem V.2.2. Residue Theorem.

Let f be analytic in the region G, except for the isolated singularities $a_1, a_2, \ldots a_m$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G then

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{m} n(\gamma; a_k)\,\text{Res}(f; a_k).$$

Note. The Residue Theorem allows us to evaluate certain integrals, provided we can evaluate winding numbers and residues. The following result allows us to compute residues in terms of derivatives.

Proposition V.2.4. Suppose f has a pole of order m at $z = a$. Let $g(z) = (z-a)^m f(z)$. Then

$$\text{Res}(f;a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

Note V.2.A. If $z = a$ is a simple pole of f, then

$$f(z) = \frac{a_{-1}}{z-a} + \sum_{k=0}^{\infty} a_k (z-a)^k \text{ and } \text{Res}(f;a) = \lim_{z\to a}(z-a)f(z) = a_{-1}.$$

Example V.2.5. Show $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}$.

Solution. With $f(z) = z^2/(1+z^4)$, f has simple poles at the 4th roots of -1, $a_1 = \exp(\pi i/4)$, $a_2 = \exp(3\pi i/4)$, $a_3 = \exp(5\pi i/4)$, and $a_4 = \exp(7\pi i/4)$. So by Note V.2.A,

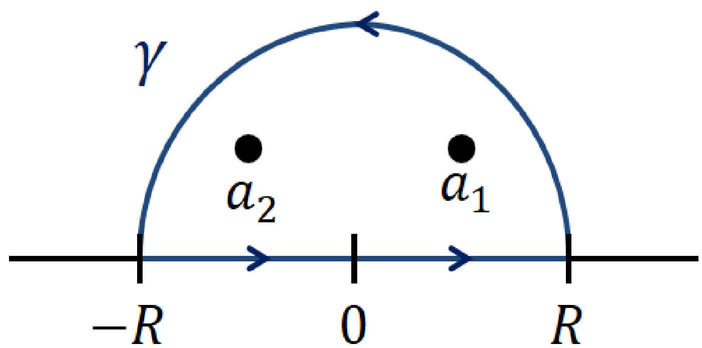
$$\begin{aligned}
\text{Res}(f;a_1) &= \lim_{z\to a_1}(z-a_1)f(z) = \lim_{z\to a_1}(z-a_1)\frac{z^2}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)} \\
&= \frac{a_1^2}{((a_1-a_2)(a_1-a_3)(a_1-a_4)} = \frac{i}{(2/\sqrt{2})(2/\sqrt{2}+2i/\sqrt{2})(2i/\sqrt{2})} \\
&= \frac{2\sqrt{2}}{4i(2+2i)}\left(\frac{2-2i}{2-2i}\right) = \frac{4\sqrt{2}(1-i)}{4i8} = \frac{1-i}{4\sqrt{2}},
\end{aligned}$$

and

$$\text{Res}(f;a_2) = \lim_{z\to a_2}(z-a_2)\frac{z^2}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)}$$

$$= \frac{a_2}{(a_2 - a_1)(a_2 - a_3)(a_2 - a_4)} = \frac{-i}{(-2/\sqrt{2})(2i/\sqrt{2})(-2/\sqrt{2} + 2i/\sqrt{2})}$$
$$= \frac{-2\sqrt{2}i}{(-2)(2i)(-2+2i)} = \frac{\sqrt{2}}{-4(1-i)}\left(\frac{1+i}{1+i}\right) = \sqrt{2}(1+i) - 8 = \frac{-1-i}{4\sqrt{2}}.$$

Let $R > 1$ and let γ be the closed path:



Then by the Residue Theorem (Theorem V.2.2),

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \mathrm{Res}(f; a_1) + \mathrm{Res}(f; a_2) = \frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}} = \frac{-i}{2\sqrt{2}}.$$

But breaking γ into the interval $[-R, R] \subset \mathbb{R}$ and the semicircle $\{Re^{it} \mid 0 \le t \le \pi\}$ gives

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx + \frac{1}{2\pi i}\int_0^\pi \frac{R^2 e^{2it}}{1+R^4 e^{4it}} iRe^{it}\,dt$$

and so

$$\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}} - iR^3 \int_0^\pi \frac{e^{3it}}{1+R^4 e^{4it}}\,dt.$$

For $0 \le t \le \pi$, $|1 + R^4 e^{4it}| \ge R^4 - 1$, so

$$\left| iR^3 \int_0^\pi \frac{3e^{it}}{1+R^4 e^{4it}}\,dt \right| \le \frac{\pi R^3}{R^4 - 1}.$$

So

$$\lim_{R\to\infty} \left(iR^3 \int_0^\pi \frac{e^{3it}}{1+R^4 e^{4it}}\,dt \right) = 0$$

and

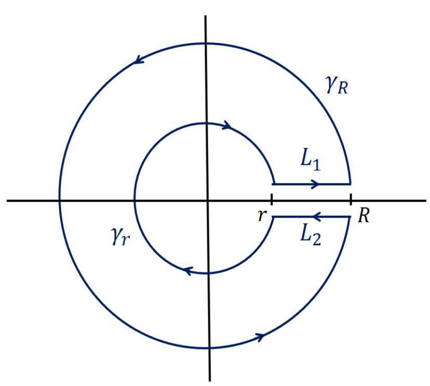
$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx\right) = \lim_{R\to\infty}\left(\frac{\pi}{\sqrt{2}} - iR^3\int_0^{\pi} \frac{e^{3it}}{1+R^4e^{4it}}\,dt\right) = \frac{\pi}{\sqrt{2}}.$$

Notice that this was evaluated using residues and no mention is made of antiderivatives!

Example V.2.12. Show that $\displaystyle\int_0^{\infty} \frac{x^{-c}}{1+x}\,dx = \frac{\pi}{\sin(\pi c)}$ for $0 < c < 1$.

Solution. We want to use residues and circles around 0 to set up a contour integral that produces the desired real definite integral in the limit. However, $z^{-c} = \exp(-c\log z)$ requires a branch of the logarithm and so a branch cut from 0 to ∞.

Let $G = \{z \mid z \neq 0 \text{ and } 0 < \arg(z) < 2\pi\}$. Define a branch of the logarithm on G of $\ell(z) = \ell(re^{i\theta}) = \log(r) + i\theta$ where $0 < \theta \leq 2\pi$. Then on G, $f(z) = \exp(-c\ell(z))$ is a branch of z^{-c}. Now we define contour γ over which we will integrate. Let $0 < r < a < R$ and let $\delta > 0$. Let L_1 be the line segment $[r+\delta i, R+\delta i]$, let γ_R be the part of the circle $|z| = R$ from $R+\delta i$ counterclockwise to $R-\delta i$, let L_2 be the line segment $[R-\delta i, r-\delta i]$, and let γ_r be the part of the circle $|z| = r$ from $r-\delta i$ clockwise to $r+\delta i$. Put $\gamma = L_1 + \gamma_R + L_2 + \gamma_r$. See figure the figure below.



Then $\{\gamma\} \subset G$, $\gamma \sim 0$ in G, and -1 is inside γ. Now $z^{-c}/(1+z)$ has a simple pole at $z = -1$ so by Note V.2.A,

$$\text{Res}(z^{-c}/(1+z); -1) = \lim_{z \to -1} (1+z)(z^{-c}/(1+z)) = \lim_{z \to -1} z^{-c}$$

$$= f(-1) = \exp(-c(\log(1) + i(\pi))) = e^{-ci\pi}.$$

By the Residue Theorem (Theorem V.2.2),

$$\int_\gamma \frac{z^{-c}}{1+z}\,dz = 2\pi i \text{Res}(z^{-c}/(1+z)) = 2\pi i e^{-ci\pi}.$$

Now $L_1 = [r + \delta i, R + \delta i]$ can be parameterized as $L_1(t) = t + \delta i$ for $t \in [r, R]$. Then with $f(z) = z^{-c}$,

$$\int_{L_1} \frac{f(z)}{1+z}\,dz = \int_r^R \frac{f(t+i\delta)}{1+t+i\delta}\,dt.$$

To deal with this integral, we define $g(t, \delta)$ on compact set $[r, R] \times [0, \pi/2]$ as

$$g(t,\delta) = \begin{cases} \left| \frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right| & \text{if } \delta \in (0, \pi/2] \\ 0 & \text{if } \delta = 0. \end{cases}$$

Then g is continuous and so, by Theorem II.5.15, uniformly continuous. So if $\varepsilon > 0$ then there is $\delta_0 > 0$ such that if $(t-t')^2 + (\delta-\delta')^2 < \delta_0^2$ then $|g(t,\delta) - g(t',\delta')| < \varepsilon/R$. In particular, with $t = t'$ and $\delta' = 0$, we have $g(t', \delta') = g(t', 0) = 0$ and so for $(t-t')^2 + (\delta - \delta') = \delta^2 < \delta_0^2$ (or $\delta < \delta_0$), $|g(t,\delta) - g(t',\delta')| = g(t,\delta) < \varepsilon/R$. So for $\delta < \delta_0$ we have $\int_r^R g(t,\delta)\,dt \le (\varepsilon/R)R = \varepsilon$. So $\lim_{\delta \to 0^+} \int_r^R g(t,\delta)\,dt = 0$ and

$$\lim_{\delta \to 0^+} \left| \int_r^R \frac{f(t+i\delta)}{1+t+i\delta}\,dt - \int_r^R \frac{t^{-c}}{1+t}\,dt \right| = \lim_{\delta \to 0^+} \left| \int_r^R \left(\frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right) dt \right|$$

$$\le \lim_{\delta \to 0^+} \left(\int_r^R \left| \frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right| dt \right) = \lim_{\delta \to 0^+} \left(\int_r^R g(t,\delta)\,dt \right) = 0,$$

and so

$$\lim_{\delta\to 0^+}\left(\int_r^R \frac{f(t+i\delta)}{1+t+i\delta}\,dt\right) = \int_r^R \frac{t^{-c}}{1+t}\,dt \text{ or } \lim_{\delta\to 0^+}\left(\int_r^R \frac{f(z)}{1+z}\,dz\right) = \int_r^R \frac{t^{-c}}{1+t}\,dt.$$

Similarly, as is to be shown in Exercise V.2.A,

$$-e^{-2\pi i}\int_r^R \frac{t^{-c}}{1+t}\,dt = \lim_{\delta\to 0^+}\int_{L_2}\frac{f(z)}{1+z}\,dz.$$

As shown above, $\displaystyle\int_\gamma \frac{f(z)}{1+z}\,dz = 2\pi i e^{-\pi c}$ and this is independent of δ so letting $\delta \to 0^+$ we have

$$2\pi i e^{-i\pi c} = \lim_{\delta\to 0^+}\left(\int_\gamma \frac{f(z)}{1+z}\,dz\right) = \lim_{\delta\to 0^+}\left(\int_{L_1}\frac{f(z)}{1+z}\,dz + \int_{\gamma_R}\frac{f(z)}{1+z}\,dz\right.$$

$$\left. + \int_{L_2}\frac{f(z)}{1+z}\,dz + \int_{\gamma_r}\frac{f(z)}{1+z}\,dz\right)$$

$$= \int_r^R \frac{t^{-c}}{1+t}\,dt - e^{-i\pi c}\int_r^R \frac{t^{-c}}{1+t}\,dt + \lim_{\delta\to 0^+}\left(\int_{\gamma_r}\frac{f(z)}{1+z}\,dz + \int_{\gamma_R}\frac{f(z)}{1+z}\,dz\right). \qquad (2.16)$$

Now if $\rho > 0$ and $\rho \neq 1$ and if γ_ρ is the part of the circle $|z| = \rho$ from $\sqrt{\rho^2-\delta^2}+i\delta$ to $\sqrt{\rho^2-\delta^2}-i\delta$ then

$$\left|\int_{\gamma_\rho}\frac{f(z)}{1+z}\,dz\right| \le \int_{\gamma_\rho}\frac{|f(z)|}{|1+z|}\,|dz| \le \frac{\rho^{-c}}{|1-\rho|}2\pi\rho.$$

Since this bound is independent of δ, from (2.16), we have

$$\left|2\pi i e^{-i\pi c} - (1-e^{-i\pi c})\int_r^R \frac{t^{-c}}{1+t}\,dt\right| \le \frac{r^{-c}}{|1-r|}2\pi r + \frac{R^{-c}}{|1-R|}2\pi R.$$

Now taking limits $r \to 0^+$ and $R \to \infty$, $\dfrac{r^{-c}}{|1-r|}2\pi r \to 0$ and $\dfrac{R^{-c}}{|1-R|}2\pi R \to 0$ since $0 < c < 1$. Hence

$$(1-e^{-2\pi i c})\int_0^\infty \frac{t^{-c}}{1+t}\,dt = 2\pi i e^{-i\pi c}$$

or

$$\int_0^\infty \frac{t^{-c}}{1+t}\,dt = \frac{2\pi i e^{-i\pi c}}{1-e^{-2i\pi c}} = \frac{2\pi i}{e^{i\pi c}-e^{-i\pi c}} = \frac{2\pi i}{2i\sin(\pi c)} = \frac{\pi}{\sin(\pi c)}$$

since $\sin z = (e^{iz} - e^{-iz})/(2i)$.

Note. A much easier solution to the previous example is given in *Schaum's Outline Series, Complex Variables* by Murray Spiegel [1964], page 185. Unfortunately, it is incorrect! Effectively, Spiegel takes $\delta = 0$ in our notation. But then γ_r and γ_R both must contain points on the branch cut of the logarithm (and hence on the branch cut of z^{-c}).

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