V.3. The Argument Principle

Note. In this section, we concentrate on zeros and poles of a function. In the Argument Principle we relate the value of an integral to winding numbers of zeros or poles. In Rouche's Theorem, a quantity related to the number of zeros and the number of poles is given which is preserved between functions satisfying a certain (inequality) relationship. The specific class of functions of concern is defined in the following.

Definition V.3.3. If G is open and f is a function defined and analytic on G except for poles, then f is a *meromorphic function* on G.

Note. If f is meromorphic on G, then we can define $f : G \to \mathbb{C}_\infty$ by setting $f(z) = \infty$ at each pole of f. By Exercise V.3.4 f is then a continuous mapping where we treat $\mathbb{C}_\infty$ as a metric space with the metric given in section I.6.

Note. If f is analytic at $z = a$ and f has a zero of order m at $z = a$, then $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$ be Definition IV.3.1. Hence

$$\frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1}g(z) + (z-a)^m g'(z)}{(z-a)^m g(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}. \tag{3.1}$$

Since $g(a) \neq 0$, then g'/g is analytic "near" $z = a$.

Note. If f has a pole of order m at $z = a$, then $f(z) = (z-a)^{-m}g(z)$ where g is analytic at $z = a$ and $g(a) \neq 0$ by the definition of pole of order m and Proposition V.1.6. Then

$$\frac{f'(z)}{f(z)} = \frac{-m(z-a)^{-m-1}g(z) + (z-a)^{-m}g'(z)}{(z-a)^{-m}g(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)}. \qquad (3.2)$$

Again, since $g(a) \neq 0$, then g'/g is analytic "near" $z = a$.

Theorem V.3.4. Argument Principle.

Let f be meromorphic in G with poles $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$ repeated according to multiplicity. If γ is a closed rectifiable curve in G where $\gamma \approx 0$ and not passing through $p_1, p_2, \ldots, p_m, z_1, z_2, \ldots, z_n$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).$$

Note. Given the representation of f'/f given in the proof, we see that winding numbers naturally arise here. Also, we would expect a primitive of f'/f to be $\log(f)$, which of course does not exist on $\{\gamma\}$ (unless the winding numbers are 0), but again this hints at multiples of $2\pi i$.

Theorem V.3.6. Let f be meromorphic on region G with zeros $z_1, z_2, \ldots, z_n$ and poles $p_1, p_2, \ldots, p_m$ repeated according to multiplicity. If g is analytic on G and γ is a closed rectifiable curve in G where $\gamma \approx 0$ and γ does not pass through any zero or pole of f, then

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} g(z_k)n(\gamma; z_k) - \sum_{j=1}^{m} g(p_j)n(\gamma; p_j).$$

Note. The proof of Theorem V.3.6 is to be given in Exercise V.3.1.

Proposition V.3.7. Let f be analytic on an open set containing $\overline{B}(a;R)$ and suppose that f is one to one on $B(a;R)$. If $\Omega = f[B(a;R)]$ and γ is the circle $|z-a| = R$, then $f^{-1}(\omega)$ is defined for each $\omega \in \Omega$ by

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)-\omega}\,dz.$$

Note. We now state as "Rouche's Theorem" what is actually a generalization of the traditional version (see Conway's reference at the bottom of page 125).

Theorem V.3.8. Rouche's Theorem.
Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$. Suppose Z_f and Z_g are the number of zeros inside γ, and P_f and P_g are the number of poles inside γ (counted according to their multiplicities) and that $|f(z)+g(z)| < |f(z)|+|g(z)|$ on γ. Then $Z_f - P_f = Z_g - P_g$.

Note. Rouche's Theorem can be further generalized by replacing the circle $\gamma = \{z \mid |z-a| = R\}$ with any closed rectifiable curve γ where $\gamma \approx 0$ in G, and with the introduction of winding numbers (this is Exercise V.3.7).

Note. Ahlfors in his *Complex Analysis* (McGraw Hill, 1979, page 153) state Rouche's Theorem as:

Let $\gamma \approx 0$ in region G where $n(\gamma; z)$ is either 0 or 1 for any point $z \neq \{\gamma\}$. Let f and g be analytic in G and for all $z \in \{\gamma\}$ suppose $|f(z) - g(z)| < |f(z)|$. Then f and g have the same number of zeros enclosed by γ.

Note. Another statement of Rouche's Theorem (see page 119 of Murray Spiegel, *Complex Variables with an Introduction to Conformal Mapping and Its Applications* in the Schaum's Outline Series, NY: McGraw-Hill, 1964):

If f and g are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

This version follows from Ahlfors' version by replacing Ahlfors' $g(z)$ with $f(z) + g(z)$.

Note. Rouche's Theorem can be used to give another easy (analytic) proof of the Fundamental Theorem of Algebra.

Theorem. Fundamental Theorem of Algebra.

If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$ is a (complex) polynomial of degree n, then p has n zeros (counting multiplicities).

Proof. We have

$$\frac{p(z)}{z^n} = 1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_2}{z^{n-2}} + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}$$

for $z \neq 0$, and $\lim_{z\to\infty} \frac{p(z)}{z^n} = 1$. So with $\varepsilon = 1$, we have that there exists $R > 0$ such that for all $|z| > R$ we have $\left|\frac{p(z)}{z^n} - 1\right| < \varepsilon = 1$. That is, for $|z| > R$, $|p(z) - z^n| < |z^n|$. With $f(z) = z^n$ and $g(z) = p(z)$, we have by Ahlfor's version of Rouche's Theorem (of course, this also follows from Conway's version as well) that, since $f(z) = z^n$ has n zeros, then $g(z) = p(z)$ has the same number of zeros. ■

Revised: 5/1/2018