# Chapter VI. The Maximum Modulus Theorem

## **VI.1.** The Maximum Principle

**Note.** We pick up where we left off in section IV.3 and introduce several more versions of the Maximum Modulus Theorem.

**Note.** Recall the original version:

**Theorem IV.3.11. Maximum Modulus Theorem.**

If $G$ is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then $f$ is constant.

**Theorem VI.1.2. Maximum Modulus Theorem—Second Version.**

Let $G$ be a bounded open set in $\mathbb{C}$ and suppose $f$ is a continuous function on $\overline{G}$ which is analytic in $G$. Then

$$\max\{|f(z)| \mid z \in \overline{G}\} = \max\{|f(z)| \mid z \in \partial G\}.$$

($\overline{G}$ is $G$ closure and $\partial G$ is the boundary of $G$.)

**Note.** The first version of the Maximum Modulus Theorem applies to an open connected set, and the second version applies to a bounded connected set. Boundedness is necessary in the Maximum Modulus Theorem—Second Version as can be seen by considering set $G = \{z \mid \text{Im}(z) \in (-\pi/2, \pi/2)\}$ and $f(z) = \exp(\exp z)$. Then $f$ is entire and so analytic and continuous, as needed. If $z \in \partial(G)$ then $z = x \pm i\pi/2$ where $x \in \mathbb{R}$ and

$$|f(z)| = |\exp(\exp(x \pm i\pi/2))| = |\exp(\pm ie^x)| = 1.$$

But 1 is not a bound since $f$ is unbounded on the real axis ($f(x) = \exp(\exp x)$). We can get a third version of the Maximum Modulus Theorem by modifying the boundary concept.

**Definition.** If $f : G \to \mathbb{R}$ and $a \in \overline{G}$ or $a = \infty$ then the *limit superior of $f(z)$ as $z$ approaches $a$* is

$$\limsup_{z \to a} f(z) = \lim_{r \to 0^+} \left( \sup\{f(z) \mid z \in G \cap B(a; r)\} \right).$$

(If $a = \infty$, $B(a; r)$ is the ball in the metric of $\mathbb{C}_\infty$ encountered in section I.6.) Similarly, define

$$\liminf_{z \to a} f(z) = \lim_{r \to 0^+} \left( \inf\{f(z) \mid z \in G \cap B(a; r)\} \right).$$

**Definition.** If $G \subseteq \mathbb{C}$ then let $\partial_\infty(G)$ denote the boundary of $G$ in $\mathbb{C}_\infty$, called the extended boundary of $G$. (Notice that for bounded $G$, $\partial_\infty(G) = \partial(G)$, and for unbounded $G$ $\partial_\infty(G) = \partial(G) \cup \{\infty\}$.)

## Theorem VI.1.4. Maximum Modulus Theorem—3rd Version.

Let $G$ be a region in $\mathbb{C}$ and $f$ an analytic function on $G$. Suppose there is a constant $M$ such that $\limsup\limits_{z \to a} |f(z)| \leq M$ for all $a$ in $\partial_\infty(G)$. Then $|f(z)| \leq M$ for all $z$ in $G$.

**Note.** For $f(z) = \exp(\exp z)$ and $G$ as in the above example, $\infty \in \partial_\infty(G)$ and $\limsup\limits_{z \to \infty} |f(z)| = \infty$ (let $z$ be real, positive and $z \to \infty$). So the Maximum Modulus Theorem—3rd Version does not apply.

**Note.** We'll see below that the hypothesis on the behavior of $f$ "at $\infty$" can be weakened (see Theorem VI.1.D).

**Note.** Two simpler versions of the maximum Modulus Theorem—3rd Version (sometimes called the Maximum Modulus Theorem for Unbounded Domains) are the following.

## Theorem VI.1.A. Maximum Modulus Theorem for Unbounded Domains (Simplified 1).

Let $R > 0$ and suppose $f$ is analytic on the complement of $\overline{B}(a; R)$, continuous on $\mathbb{C} \setminus B(a; R)$, $|f(z)| \leq M$ on $\partial(B(a; R))$, and $\lim\limits_{|z| \to \infty} |f(z)| \leq M$. Then $|f(z)| \leq M$ on $\mathbb{C} \setminus B(a; R)$.

**Theorem VI.1.B. Maximum Modulus Theorem for Unbounded Domains (Simplified 2).**

Let $R > 0$ and suppose $f$ is analytic on the complement of $\overline{B}(a; R)$. Suppose $\lim_{z \to a, |z| > R} |f(z)| \leq M$ for all $|a| = R$ and $\lim_{z \to \infty} |f(z)| \leq M$. Then $|f(z)| \leq M$ on $\mathbb{C} \setminus \overline{B}(a; R)$.

**Note.** This simplified versions are often useful in applications involving polynomials.

**Note.** Exercise VI.1.1 requests a proof of the following:

> **Theorem VI.1.C. Minimum Modulus Theorem.** If $f$ is a non-constant analytic function on a bounded open set $G$ and is continuous on $G^-$, then either $f$ has a zero in $G$ or $|f|$ assumes its minimum value on $\partial(G)$.

**Note.** In the Maximum Modulus Theorem—3rd Version (Theorem VI.1.4), if $\infty \in \partial_\infty(G)$ then we required that $\lim_{z \to \infty} |f(z)| \leq M$. In fact, this condition can be weakened. The following is Corollary 1.6.13 from Q. I. Rahman and G. Schemisser's *Analytic Theory of Polynomials*, London Mathematical Society Monographs (Book 26), Clarendon Press (2002).

## Theorem VI.1.D. Maximum Modulus Theorem for Unbounded Domains 2.

Let $z(t)$, $t \in [\alpha, \beta]$, define a Jordan curves $\Gamma$ with its trace in $\mathbb{C}$ (that is, $\Gamma$ is a simple closed curve in $\mathbb{C}$), and denote the open interior of $\Gamma$ by $\Omega$. Also, let $\varphi$ be a function which is analytic in $\mathbb{C} \setminus \{\Gamma \cup \Omega\}$ and continuous on $\mathbb{C} \setminus \Omega$ such that $|\varphi(z)| \leq M$ for all $z \in \Gamma$. Suppose, in addition, that $\varphi(z)$ tends to a finite limit $\ell$ as $z$ tends to infinity and set $\varphi(\infty) = \ell$. Then, $|\varphi(z)| \leq M$ for all $x$ in $\mathbb{C}_\infty \setminus \{\Gamma \cap \Omega\}$, unless $\varphi$ is a constant.

**Note.** Of course the hypotheses of the simplified versions of the Maximum Modulus Theorem for Unbounded Domains (Theorems VI.1.B and VI.1.C) can now be weakened by replacing "$\lim_{z \to \infty} |f(z)| \leq M$" with "$\lim_{z \to \infty} |f(z)|$ is finite."

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