VI.2. Schwarz's Lemma

Note. Schwarz's Lemma is an inequality concerning functions on the unit disk. As an inequality, I have used it in my research, and we will see some generalizations.

Lemma VI.2.1. Schwarz's Lemma.
Let $D = \{z \mid |z| < 1\}$ and suppose f is analytic on D with
(a) $|f(z)| \leq 1$ for $z \in D$, and
(b) $f(0) = 0$.
Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all z in the disk D. Moreover if $|f'(z)| = 1$ of if $|f(z)| = |z|$ for some $z \neq 0$ then there is a constant $c \in \mathbb{C}$, $|c| = 1$, such that $f(w) = cw$ for all $w \in D$.

Note. A proof of the following version of Schwarz's Lemma is similar to the proof of Lemma VI.2.1. We leave the proof as Exercise 6.2.A.

Lemma VI.2.A. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ for $|z| \leq R$. Suppose $f(0) = 0$. Then

$$|f(z)| \leq \frac{M|z|}{R} \text{ for any } |q| \leq R.$$

Note. We now classify conformal maps of the open unit disk D onto itself.

Definition. For $|a| < 1$, define the Möbius transformation

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

Note. φ_1 is analytic for $|z| < 1/|\overline{a}| = 1/|a|$, and so φ_a is analytic on the unit disk D (since $1/|a| > 1$). Also,

$$\varphi_a(\varphi_{-a}(z)) = \varphi_a\left(\frac{z+a}{1+\overline{a}z}\right) = \frac{(z+a)/(1+\overline{a}z) - a}{1 - \overline{a}(z+a)/(1+\overline{a}z)}$$

$$= \frac{(z+a) - a(1+\overline{a}z)}{(1+\overline{a}z) - \overline{a}(z+a)} = \frac{z(1-|a|^2)}{a - |a|^2} = z,$$

and similarly $\varphi_{-a}(\varphi_a(z)) = z$. This is valid for all $z \in D$, so φ_a is one to one on D.

Proposition VI.2.2. If $|a| < 1$ then φ_a is a one to one map of the open unit disk D onto itself. The inverse of φ_a is φ_{-a}. Furthermore, φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a'(0) = 1 - |a|^2$, and $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Lemma IV.2.A. Suppose f is analytic on $D = \{z \mid |z| < 1\}$ and $|f(z)| \le 1$ for $z \in D$. Let $z \in D$. Then

$$|f'(a)| \le \frac{1 - |f(a)|^2}{1 - |a|^2}.$$

Moreover, equality holds exactly when $f(z) = \varphi_\alpha(c\varphi_a(z))$, where $\alpha = f(a)$ for some $c \in \mathbb{C}$ where $|c| = 1$.

Theorem VI.2.5. Let $f : D \to D$ be a one to one analytic map of D onto itself and suppose $f(a) = 0$. Then there is a complex c where $|c| = 1$ such that $f = c\varphi_a$.

Note. Proposition 2.2 and Theorem 2.5 combine to show that the one to one analytic maps of $D = \{z \mid |z| < 1\}$ onto itself are precisely the functions of the form $f(z) = c\left(\frac{z-a}{1-\overline{a}z}\right)$ where $|z| = 1$ and $|a| < 1$. Notice that

$$f'(z) = v\frac{[1](1-\overline{a}z) - (z-a)[-\overline{a}]}{(1-\overline{a}z)^2} = c\frac{1-|a|^2}{(1-\overline{a}z)^2}$$

and so $f'(z) \neq 0$ for $z \in D$ and by Theorem III.3.4 these mappings are all conformal on D.

Note. Schwarz's Lemma gives us a bound for $|f(z)|$ pointwise on $D = \{z \| z \| < 1\}$; namely, $|f(z)| \leq |z|$. The hypotheses are $|f(z)| \leq 1$ for $z \in D$ and $f(0) = 0$. The following is a slight generalization of this version of Schwarz's Lemma. It is stated on $\overline{D} = \{z \mid |z| \leq 1\}$. Conway's version of Schwarz's Lemma holds with D replaced by $\overline{D}$.

Generalized Schwarz's Lemma 1. If f is analytic on $\overline{D}\{z \mid |z| \leq 1\}$, with

(a) $|f(z)| \leq M$ for $z \in \overline{D}$, and

(b) $f(a) = 0$ where $|a| < 1$.

Then for $z \in \overline{D}$:

$$|f(z)| \leq M\left|\frac{z-a}{a-\overline{a}z}\right| = M|\varphi_a(z)|.$$

Note. The above generalization of Schwarz's Lemma is really just a matter of book keeping. That is, we shifted the unique zero of f from 0 to a and modified the bound on $|f(z)|$ from 1 to the more general M. However, there is not really anything conceptually new in the result. The following generalization of the original version. It uses information about $f(0)$ (assumed to be 0 in the original version) <u>and</u> information about $f'(0)$ (think of f as having a series representation and this result uses information about both the constant term and the z coefficient).

Generalized Schwarz's Lemma 2. (From N. K. Govil, Q. Rahman, and G. Schmeisser, On the Derivative of a Polynomial, *Illinois Journal of Math*, **23** (1979), 319–329.) If f is analytic on $\overline{D} = \{z \mid |z| \le 1\}$ with

(a) $|f(z)| \le 1$ for $z \in \overline{D}$,

(b) $f(0) = a$, and

(c) $f'(0) = b$.

Then for $x \in \overline{D}$

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.$$

This result is best possible ("sharp") and equality holds for

$$f(z) = \frac{a + (b/(a+a)z - z^2}{1 - (b/(1+a))z - az^2}.$$

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