VI.3. Convex Functions and Hadamard's Three Circles Theorem

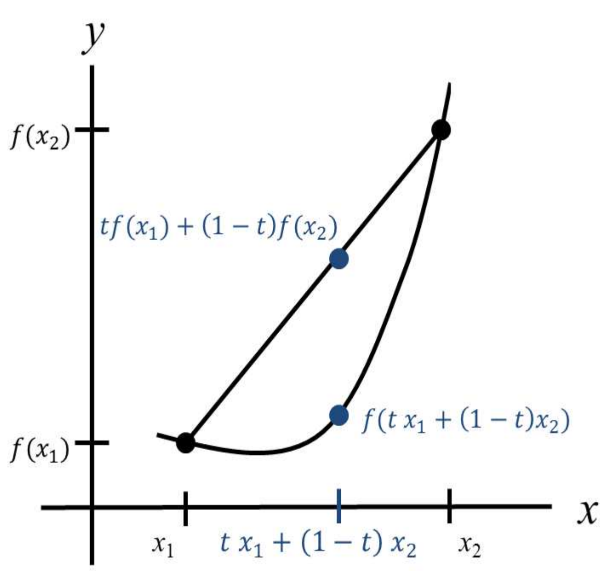
Note. This section relates to "rates of growth" results and indicates how large a function $f(z)$ can get (in modulus) in terms of the real part of the input variable z and in terms of the modulus of z.

Definition VI.3.1. If $[a,b] \subset \mathbb{R}$, then a function $f : [a,b] \to \mathbb{R}$ is *convex* if for any $x_1, x_2 \in [a,b]$ we have

$$f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1)$$

for all $t \in [0,1]$. A set $A \subset \mathbb{C}$ is *convex* if whenever $z, w \in A$ then $tz + (1-t)w \in A$ for all $t \in [0,1]$.

Note. $f : [a,b] \to \mathbb{R}$ is convex if it is (in the verbiage of Calculus 1) "concave up":



It is common to call "concave down" functions "concave." A set $A \subset \mathbb{C}$ is convex if for all points $z, w \in A$, the line segment $[z, w] \subset A$. Of course, we can relate the concept of a convex subset of $\mathbb{C}$ to the concept of a convex subset of $\mathbb{R}^2$ (and vice versa). The following result relates convex functions and convex sets.

Proposition VI.3.2. A function $f : [a, b] \to \mathbb{R}$ is convex if and only if the set

$$A = \{(x, y) \mid x \in [a, b] \text{ and } f(x) \leq y\}$$

is convex.

Note. The following follows from the definition of convex function and convex set by using math induction.

Proposition VI.3.3.

(a) A function $f : [a, b] \to \mathbb{R}$ is convex if and only if any points $x_1, x_2, \ldots, x_n \in [a, b]$ and real numbers $t_1, t_2, \ldots, t_n \geq 0$ with $\sum k = 1^n t_k = 1$, we have

$$f\left(\sum_{k=1}^{n} t_k x_k\right) \leq \sum_{k=1}^{n} t_k f(x_k).$$

(b) A set $A \subset \mathbb{C}$ is convex if and only if any points $z_1, z_2, \ldots, z_n \in A$ and real numbers $t_1, t_2, \ldots, t_n \geq 0$ with $\sum_{k=1}^{n} t_k = 1$, we have $\sum_{k=1}^{n} t_k z_k \in A$.

Proposition VI.3.4. A differentiable function f on $[a, b]$ is convex if and only if f' is increasing.

Definition. Positive valued function f is *logarithmically convex* if $\log(f(x))$ is convex.

Note. If f is logarithmically convex, then it is convex, which is easy to see for twice differentiable functions:

$$\frac{d^2}{dx^2}[\log(f(x)] = \frac{d}{dx}\left[\frac{f'(x)}{f(x)}\right] = \frac{f''(x)f(x) - f'(x)f'(x)}{(f(x))^2} \geq 0$$

$$\Rightarrow f''(x) \geq \frac{(f'(x))^2}{f(x)} \geq 0 \text{ (since } f(x) > 0).$$

So $\log(f(x))$ is convex implies that $f(x)$ is convex by Proposition 3.4.

Theorem VI.3.7. Let $a < b$ and let G be the vertical strip $\{x + iy \mid a < x < b\}$. Suppose $f : G^- \to \mathbb{C}$ is continuous and f is analytic in G. If we define $M : [a, b] \to \mathbb{R}$ by

$$M(x) = \sup\{|f(x + iy)| \mid -\infty < y < \infty\}$$

and $|f(z)| < B$ for all $z \in G$, then $\log M(x)$ is a convex function.

Note. We need a preliminary result before the proof of Theorem 3.7.

Lemma VI.3.10. Let f and G be as Theorem 3.7 and further suppose that $|f(z)| \leq 1$ for $z \in \partial G$. Then $|f(z)| \leq 1$ for $z \in G$.

Note. Now for the proof of Theorem 3.7.

Corollary VI.3.9. Let $a < b$ and let G be the vertical strip $\{x + iy \mid a < x < b\}$. Let $f : G^- \to \mathbb{C}$ be continuous and let f be analytic on G. Then for all $z \in G$ we have

$$|f(z)| < \sup\{|f(z)| \mid z \in \partial G\}.$$

Note. Corollary 3.9 shows how Theorem 3.7 is really a type of Maximum Modulus Theorem.

Note. Consider the annulus $A = \text{ann}(0; R_1, R_2)$ where $0 < R_1 < R_2 < \infty$, and the vertical strip $G = \{x + iy \mid \log R_1 < x < \log R_2\}$. Then the exponential function maps G onto A and ∂G onto ∂A (of course, because of the periodic nature of the exponential function, the mapping is not one to one...it is "∞ to 1"). This mapping is useful in proving the following.

Theorem VI.3.13. Hadamard's Three Circles Theorem.
Let $0 < R_1 < R_2 < \infty$ and suppose f is analytic and not identically zero on $\text{ann}(0; R_1, R_2)$. If $R_1 < r < R_2$, define

$$M(r) = \max\{|f(re^{i\theta})| \mid 0 \leq \theta \leq 2\pi\}.$$

Then for $R_1 < r_1 \leq r \leq r_2 < R_2$ and $r_1 \neq r_2$, we have

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2).$$

Note. Comparing Hadamard's Three Circles Theorem to equation $(*)$ in the proof of Theorem 3.7 (also, see line 7 of page 136), shows that Hadamard's Three Circles Theorem implies that $\log M(x)$ is a convex function of $\log x$.

Note. Of course the title "Three Circles Theorem" comes from the fact that the result compares the logarithm of the maximum modulus of a function in three circles of different radii (r_1, r_2, and r). Such results are common in classical complex analysis and are sometimes called "rate of growth results."

Revised: 8/11/2017